Finite group actions on Picard stacks

1. What is a Picard stack? The basics.

2. Duality and FM transform.

3. Finite group actions, invariants and coinvariants.

4. Ramified coverings of curves.

5. Hitchin systems and beyond.

---

§1. What is a Picard stack?

- Picard groupoids

A Picard groupoid $(P, \otimes)$ is a $\underbrace{}^{\text{(small)}}$ groupoid P

endowed with a functor

$$\otimes : P \times P \longrightarrow P$$

$$(x, y) \longmapsto x \otimes y$$

which is :

1. associative and strictly commutative :

$\exists$ isos. of functors $\sigma_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$

$\tau_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x$

(1.a) pentagon axiom.

(1.b) $\tau_{x,x} = id_x$.

(1.c) $\tau_{y,x} \circ \tau_{x,y} = id_{x \otimes y}$.

(1.d) hexagon axiom.

2. such that, $\forall x \in Ob\, P$, $T_x : y \mapsto x \otimes y$ is an equivalence.

Examples

Let G be an abelian group.

(1) Let $P = \underline{G}$ $\to$ $Obj(P) = G$

$\to$ $Hom_P(x,y) = \begin{cases} \emptyset & \text{if } x \neq y \\ \{id_x\} & \text{if } x = y. \end{cases}$

$$\otimes : P \times P \longrightarrow P$$
$$(x,y) \longmapsto x+y .$$
$$(id_x, id_y) \longmapsto id_{x+y}$$

(2) Let $P = BG \longrightarrow \mathrm{Obj}(P) = \{*\}$
$\hookrightarrow \mathrm{Hom}_P(*,*) = G$

$$\otimes : P \times P \longrightarrow P$$
$$(*,*) \longmapsto *$$
$$(x,y) \longmapsto x+y$$

(3) $P = \mathrm{Pic}(X) = \{\text{line bundles on } X\}/_{\text{iso}}$, X scheme.

← the Picard groupoid

$$\otimes : \mathrm{Pic}(X) \times \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X)$$
$$(\mathcal{L}, \mathcal{M}) \longmapsto \mathcal{L} \otimes \mathcal{M}$$

(3') $P = \mathrm{Bun}_G(X) = \{G\text{-torsors on } X\}/_{\text{iso}}$. $\otimes$ coming from $G \times G \xrightarrow{+} G$.

- Picard stacks

· Let $\mathcal{S}$ be a site ← category w. Grothendieck topology.

(typically, $\mathcal{S} = S_{fpqc}$, S noetherian scheme).

(for me, usually, $S = \mathrm{Spec}(k)$, $k = \bar{k}$).

· Stack (over $\mathcal{S}$) $\equiv$ "sheaf of groupoids"

$$\mathcal{X} : \mathcal{S} \longrightarrow \text{Groupoids}$$

- presheaf (natural functorial axioms)
- sheaf condition on internal hom
- descent ("gluing").

- A Picard stack is a stack $\mathcal{X} : \mathcal{S} \to$ Groupoids that factors through Picard Groupoids $\hookrightarrow$ Groupoids (a sheaf of Picard groupoids).

<u>Examples</u> Let G be a group object in $\mathcal{S}$.

[that is, $G \to S$ is a group scheme].

[for me, G is a group over k].

- $\underline{G}$ $\leftarrow$ the group scheme regarded as a stack (the functor of points) (as a sheaf)

$$\underline{G}(S) = \underline{G(S)} \quad \text{a Picard groupoid.}$$

- $BG = [S/G]$ $\leftarrow$ (quotient stack)

$T \to S$ $\qquad BG(T) = \{G\text{-torsors on } T\}$

- (The) <u>Relative Picard stack</u>: $\quad Pic_{X/S}$,

for $T \to S$, $\quad Pic_{X/S}(T) = \{$flat families of line bundles over X parametrized by $T\}$

Similarly: for $G \to X$ group scheme $\qquad T \to S$

$$Bun_{G/X}(T) = \{\text{flat families of } G\text{-torsors over } X \text{ param. by } T\}.$$

OBS 1 $p: X \to S$ $\quad \mathrm{Pic}_{X/S} = p_* B\mathbb{G}_{m,X}$

$$\mathrm{Bun}_{G/X} = p_* BG.$$

OBS 2 . $\quad \mathrm{Pic}_{X/S} = B\mathbb{G}_m \times \mathrm{Pic}(X)$

If X smooth projective: (over $\mathbb{k} = \mathbb{C}$)

$$0 \to \underset{\mathrm{Jac}(X)}{\underset{\|}{\mathrm{Pic}^0(X)}} \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to 0$$

from $0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$

$$\mathrm{Pic}_{X/S} = B\mathbb{G}_m \times \mathrm{Jac}(X) \times H^2(X, \mathbb{Z}).$$

· For $\dim X = 1$, $\quad \mathrm{Pic}_{X/S} = B\mathbb{G}_m \times \underbrace{\mathrm{Jac}(X)}_{\substack{\text{abelian variety} \\ \text{of dimension} = \text{genus}(X).}} \times \mathbb{Z}$

(also for $\mathrm{char}\, \mathbb{k} \neq 0$)
$\mathbb{k} = \overline{\mathbb{k}}$

· **Deligne's POV**

· Let $C^{[-1,0]}(\mathcal{Y})$ ← category of 2-term complexes $(K_{-1} \to K_0)$ of sheaves of abelian groups $/\mathcal{Y}$.

$K = (K_{-1} \to K_0) \rightsquigarrow \mathrm{pch}(K)$ ← Picard pre-stack

- OBJECTS: K_0
- MORPHISMS: Coboundaries of K

· This induces: $\quad$ Picard Stacks $\sim D^{[-1,0]}(\mathcal{Y})$

· More generally: 2-Category of Picard Stacks ~ OBJECTS: $K \in C^{[-1,0]}(\mathcal{S})$ with K_{-1} injective

$\mathcal{P} \longrightarrow \mathcal{P}^b = (K_{-1} \to K_0)$ 1- MORPHISMS: morphisms of complexes

$\mathcal{P} = ch(K) \longleftarrow K$ 2- MORPHISMS: homotopies

OBS $|\mathcal{P}| = H^0(\mathcal{P}^b)$, $Aut_0(\mathcal{P}) = H^{-1}(\mathcal{P}^b)$.

Ex. $G \to S$ group scheme

$$\underline{G}^b = (0 \to G) \quad \leftarrow G \text{ concentrated in degree } 0$$

$$(BG)^b \underset{q.iso}{\sim} G[1] = (G \to 0)$$

Under Deligne's equivalence, we have:

$$(1) \quad (f_* \mathcal{P})^b \simeq \tau_{\leq 0} Rf_*(\mathcal{P}^b)$$

$$(2) \quad \underline{Hom}(\mathcal{P}_1, \mathcal{P}_2) \simeq \tau_{\leq 0} R\underline{Hom}(\mathcal{P}_1^b, \mathcal{P}_2^b)$$

$$(3) \quad (\mathcal{P}_1 \otimes \mathcal{P}_2)^b \simeq \tau_{\leq -1}(\mathcal{P}_1^b \otimes \mathcal{P}_2^b)$$

OBS 1 Tensor - Hom adjunction is satisfied

OBS 2 We have a good notion of short exact sequence

$0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \mathcal{P}_3 \to 0$ is short exact iff

$\mathcal{P}_1^b \to \mathcal{P}_2^b \to \mathcal{P}_3^b \to \mathcal{P}_1^b[1]$ is a distinguished triangle.

• <u>Deligne's universal coefficients formula</u> (Abel–Jacobi)

$S = \operatorname{Spec}(k)$, $k = \bar{k}$

$C \to S$ smooth projective curve

$$\underline{\operatorname{Hom}}(\operatorname{Pic}_{C/S}, BG) \xrightarrow{\sim} \operatorname{Bun}_{p^*G/C}.$$

<u>Ex.</u> (1) $\quad AJ : C \longrightarrow \operatorname{Pic}_{C/S}$ $\quad$ Abel–Jacobi map

$x \longmapsto \mathcal{O}_C(x)$

$\rightsquigarrow$

$$AJ : \underline{\operatorname{Hom}}(\operatorname{Pic}_{C/S}, B\mathbb{G}_m) \longrightarrow \operatorname{Pic}_{C/S}.$$

(2) Take $G = T$ be an algebraic torus.

$T = \Lambda \otimes \mathbb{G}_m$, $\quad \Lambda = \operatorname{Hom}(\mathbb{G}_m, T)$ cocharacter lattice

$$AJ : \operatorname{Pic}_{C/k} \otimes \underline{\Lambda} \longrightarrow \operatorname{Bun}_{T/C}$$
$$L \otimes \lambda \longmapsto L \times^{\mathbb{G}_m, \lambda} T$$

---

§2. <u>Duality on Picard stacks</u>

• $$D(\mathcal{P}) = \underline{\operatorname{Hom}}(\mathcal{P}, B\mathbb{G}_m) \leftarrow \text{the } \underline{\text{dual}} \text{ Picard stack}$$

<u>OBS</u> $\quad \exists$ canonical morphism $\mathcal{P} \longrightarrow D(D(\mathcal{P}))$.

$\mathcal{P}$ is <u>dualizable</u> if it is an <u>iso</u>.

· This duality generalizes others:

- Abelian varieties:

$A \to S$ abelian scheme

$$D(A) = \underline{\mathrm{Hom}}(A, B\mathbb{G}_m) = \mathrm{Ext}^1(A, \mathbb{G}_m) = \hat{A}.$$

$\hat{A}$ ← dual abelian scheme

- Cartier duality:

· $M \to S$ finitely generated abelian group

M^* ← Cartier dual

$$M^* = \mathrm{Hom}(M, \mathbb{G}_m) = \mathrm{Spec}(\mathcal{O}_S[M]).$$

$$D(M) = \mathrm{Hom}(M, B\mathbb{G}_m) = BM^*.$$

· $G \to S$ of multiplicative type ($G = M^*$ for some M)

$G^* = \mathrm{Hom}(G, \mathbb{G}_m)$ ← character group

$$D(BG) = M.$$

OBS Abel-Jacobi $\rightsquigarrow$ $D(\mathrm{Pic}_C) = \mathrm{Hom}(\mathrm{Pic}_C, B\mathbb{G}_m) = \mathrm{Bun}_{\mathbb{G}_m} = \mathrm{Pic}_C$.
(Deligne's UCF)

Pic_C is self-dual.

· More generally:

$$D(\mathrm{Pic}_C \otimes M) = \mathrm{Hom}(\mathrm{Pic}_C \otimes M, B\mathbb{G}_m) = \mathrm{Hom}(\mathrm{Pic}_C, D(M)) = \mathrm{Hom}(\mathrm{Pic}_C, BG) = \mathrm{Bun}_{G/C}.$$

In Particular:

$$D(AJ): D(\mathrm{Bun}_{T/C}) \longrightarrow D(\mathrm{Pic}_C \otimes \Lambda) = \mathrm{Bun}_{T^\vee/C}, \quad T^\vee = \Lambda^* \leftarrow \text{the dual torus}.$$

- <u>Fourier - Mukai transform</u>

If $\mathcal{P}$ is dualizable, there is a <u>universal line bundle</u> (Poincaré)

$$\mathcal{L}_{\mathcal{P}} \longrightarrow \mathcal{P} \times D(\mathcal{P})$$

$$\begin{aligned} \Phi_{\mathcal{P}} : D^b(QCoh(\mathcal{P})) &\longrightarrow D^b(QCoh(D(\mathcal{P}))) \\ \mathcal{F} &\longmapsto Rp_{2*}(Lp_1^* \mathcal{F} \otimes \mathcal{L}_{\mathcal{P}}) \end{aligned}$$

<u>Fourier-Mukai transform</u>

- $\mathcal{P}$ is <u>good</u> (in the sense of Arinkin) if $\Phi_{\mathcal{P}}$ is an <u>iso</u>.

<u>Ex.</u>

$$\Phi : D^b(QCoh(Bun_{T/C})) \longrightarrow D^b(QCoh(Bun_{T^\vee/C}))$$

is an <u>iso</u>.

- <u>Beilinson 1-Motives</u> (A nice class of good Picard Stacks)
("Almost abelian varieties")

- $\mathcal{P}$ is a Beilinson 1-Motive if it admits a 2-step filtration

$$W_{-1} = 0 \subseteq W_0 \subset W_1 \subseteq W_2 = \mathcal{P}$$

s.t.

- $Gr_0^W = W_0 \cong BG$, for G of multiplicative type
- $Gr_1^W = W_1/W_2 \cong A \leftarrow$ abelian scheme
- $Gr_2^W = W_2/W_1 \cong M \leftarrow$ f.g. abelian group.

- **Easy locally:** $\mathcal{P} = BG \times A \times M$

$$D(\mathcal{P}) = BM^* \times \hat{A} \times G^* .$$

- Why are Beilinson 1-motives good?

  → FM for abelian varieties is an iso. (Mukai)

  → $D^b(QCoh(M)) = \bigoplus_{m \in M} K(Vect_k)$

$$D^b(QCoh(BG)) = \{G\text{-modules}\}$$

$$D^b(QCoh(M)) \longrightarrow D^b(QCoh(BG))$$
$$k_m \longmapsto (m : G \longrightarrow \mathbb{G}_m)$$

---

§ 3. Finite group actions. Invariants and coinvariants

- Γ abstract finite group

$$\rho : \Gamma \longrightarrow \underline{Aut}(\mathcal{P}) \quad \leftarrow \mathcal{P} \text{ is a } \Gamma\text{-Picard stack}$$

[**Eg.**
$$* \longmapsto \mathcal{P}$$
$$\underbrace{B\Gamma}_{\text{As a 2-category}} \longrightarrow \text{Picard Stacks}$$
]

- $q : \Gamma_S \to S$, $\quad q_* : \{\text{Picard Stacks} / \Gamma_S\} \longrightarrow \{\Gamma\text{-Pic. Stacks} / S\}$.

$u : S \to \Gamma_S$ trivial section.

$$\mathcal{P}^\Gamma := \mathrm{Hom}_{\Gamma_S}(u_* \mathbb{G}_a, \mathcal{P}) \overset{\mathrm{ch}}{=} \tau_{\leq 0} R\underline{\mathrm{Hom}}_{\mathcal{O}_S[\Gamma]}(u_* \mathcal{O}_S, \mathcal{P}^b) = \tau_{\leq 0} \mathbb{H}^\bullet(\Gamma, \mathcal{P}^b).$$

$$\mathcal{P}_\Gamma := u_* \mathbb{G}_a \otimes_{\Gamma_S} \mathcal{P} = \tau_{\geq -1} (u_* \mathcal{O}_S \otimes^L_{\mathcal{O}_S[\Gamma]} \mathcal{P}^b) = \tau_{\geq -1} \mathbb{H}_\bullet(\Gamma, \mathcal{P}^b).$$

<u>OBS</u> $$D(\mathcal{P}_\Gamma) = D(\mathcal{P})^\Gamma$$ (because of tensor-hom adjunction).

<u>Examples</u>

Let $G \to S$ be a comm. group scheme.

$$\underline{G}^\Gamma = \underline{G^\Gamma}, \quad \underline{G}_\Gamma = \underline{G_\Gamma} \times BH_1(\Gamma, G)$$

$$(BG)^\Gamma = BG^\Gamma \times H^1(\Gamma, G), \quad (BG)_\Gamma = BG_\Gamma.$$

- More generally, I can use the spectral sequence:

  - $\mathcal{P}^\Gamma \cong \mathrm{ch}((\mathcal{P}^b)^\Gamma) \leftarrow$ <u>Homotopy invariants</u>

  - $|\mathcal{P}_\Gamma| = |\mathcal{P}|_\Gamma$ (coinvariants as group).

---

§ 4. <u>Ramified coverings of curves, tori and duality</u>

k algebraically closed field

T alg. torus

$\Lambda, \Lambda^\vee$ cochar. and char. lattices

$T^\vee = \Lambda^*$ dual torus

$\pi : \tilde{C} \to C$

finite Galois cover of smooth proj. curves over k

$\Gamma = \mathrm{Gal}(\tilde{C}/C)$

<u>tame ramification</u>

$T_{\tilde{C}} : T \times \tilde{C} \to \tilde{C}$

$\pi_* T_{\tilde{C}} \to C$, $J^1 = (\pi_* T_{\tilde{C}})^W$, $J^0 = (\pi_* T_{\tilde{C}})^{W,0}$

$\mathcal{P}^1 := \mathrm{Bun}_{J^1/C}$, $\mathcal{P}^0 := \mathrm{Bun}_{J^0/C}$.

[Similarly: $\check{J}^1 = (\pi_* T^\vee_C)^W$, $\check{J}^0$, $\check{\mathcal{P}}^1$, $\check{\mathcal{P}}^0$]

<u>Galois description</u>: $\mathcal{P}^1 \hookrightarrow \mathrm{Bun}^W_{T/C}$

$\mathrm{Nm} : \mathrm{Bun}_{T/C, W} \twoheadrightarrow \mathcal{P}^0$.

<u>Norm map</u>

<u>Duality</u>

$\mathcal{P}^1 \hookrightarrow \mathrm{Bun}^W_{T/C} \hookrightarrow \mathrm{Bun}_{T/C}$

$\mathrm{Bun}_{T^\vee/C} \twoheadrightarrow \mathrm{Bun}_{T^\vee/C, W} \twoheadrightarrow \check{\mathcal{P}}^0$

$$\begin{tikzcd}
D(\mathrm{Bun}_{T/C}) \arrow[r, "AJ"] \arrow[d, two heads, "D(\pi^*)"'] & \mathrm{Bun}_{T^\vee/C} \arrow[d, two heads, "\mathrm{Nm}^\vee"] \\
D(\mathcal{P}^1) \arrow[r, "\mathfrak{D}"] & \check{\mathcal{P}}^0
\end{tikzcd}$$

<u>Thm</u> $\mathfrak{D}$ is an iso. of Picard stacks.

---

§5. <u>Hitchin systems</u>

character and cocharacter lattices

G <u>reductive group</u> $/ k$ $\rightsquigarrow$ Data: $(\Lambda, \Phi, \Lambda^\vee, \Phi^\vee)$

(char $p \nmid |W|$) roots and coroots

$G^\vee$ <u>Langlands dual group</u> $\leftsquigarrow$ Dual data: $(\Lambda^\vee, \Phi^\vee, \Lambda, \Phi)$

C smooth projective curve $/ \Bbbk$

$$\mathcal{M}(C,G) \longrightarrow \mathcal{A}(C,G)$$

$\uparrow$ Hitchin fibration

Higgs bundles

$$\mathcal{M}(C,G^\vee) \longrightarrow \mathcal{A}(C,G^\vee)$$

<u>Thm</u> (HT, DP, CZ)

$$\begin{tikzcd}
\mathcal{M}_a \subset \mathcal{M}(C,G) \arrow[dr] & & \mathcal{M}(C,G^\vee) \supset \check{\mathcal{M}}_a \arrow[dl] \\
& a \in \mathcal{A}(C,G) \cong \mathcal{A}(C,G^\vee) &
\end{tikzcd}$$

- $\mathcal{M}_a$ is a torsor under a Picard stack $\mathcal{P}_a$.
- $\check{\mathcal{M}}_a$ is a torsor under a Picard stack $\check{\mathcal{P}}_a$.

$$\check{\mathcal{P}}_a \cong D(\mathcal{P}_a).$$

<u>Idea:</u> $\exists$ $\tilde{C}_a \to C$ (<u>cameral cover</u>) s.t.

$$\mathcal{P}_a = \mathcal{P}^1$$
$$\check{\mathcal{P}}_a = \check{\mathcal{P}}^0 .$$