Higgs bundles twisted by a vector bundle

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Holomorphic vector bundles over Riemann surfaces

Let X be a compact Riemann surface of genus g.

- Smooth classification: rank and degree (1^{st} Chern class).
- Line bundles: Picard group, $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$.
  Components of fixed degree $\mathrm{Pic}^d(X)$ are isomorphic to the **Jacobian**

$$J(X) = \frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})}.$$

- $g = 0$, Grothendieck (1957).
- $g = 1$, Atiyah (1957).
- $g \geq 2$?

For now on, assume $g \geq 2$.

Stable vector bundles

Definition

A holomorphic vector bundle $E \to X$ is **stable** if for every proper holomorphic subbundle $E' \subset E$,

$$\frac{\deg E'}{\mathrm{rk} E'} < \frac{\deg E}{\mathrm{rk} E}.$$

- Notion of stability arising from Mumford's GIT.
- **Moduli space of stable vector bundles**: $\mathcal{N}(n, d)$

Holomorphic structures as Dolbeaut operators

Definition

Let $\mathbb{E}$ be a smooth vector bundle of rank n and degree d over X. A **holomorphic structure on** $\mathbb{E}$ is an operator

$$\bar{\partial}_E : \Omega^{p,q}(X, \mathbb{E}) \longrightarrow \Omega^{p,q+1}(X, \mathbb{E})$$

which satisfies

$$\bar{\partial}_E(\alpha\psi) = (\bar{\partial}\alpha)\psi + (-1)^p \alpha \wedge \bar{\partial}_E \psi.$$

- $\bar{\partial}_E^2 = 0$ is the integrability condition for the PDE $\bar{\partial}_E s = 0$.
- The solutions of $\bar{\partial}_E s = 0$ are holomorphic vector functions (locally free sheaf over $\mathcal{O}_X$).
- $E = (\mathbb{E}, \bar{\partial}_E)$ is a holomorphic vector bundle.

The gauge-theoretic point of view

$(\mathbb{E}, h)$ smooth complex Hermitian vector bundle of rank n and degree d over X.

$$h\text{-unitary connections} \longleftrightarrow \text{Holomorphic structures on } \mathbb{E}$$
$$\nabla \longmapsto \nabla^{0,1}$$

- $\mathcal{A}$ set of irreducible h-unitary connections on $(\mathbb{E}, h)$.
- $\mathcal{G}$ group of unitary gauge transformations of $\mathbb{E}$.
- $\mathcal{G}$ acts on $\mathcal{A}$ by conjugation with momentum map

$$\mu : \mathcal{A} \longrightarrow \mathrm{Lie}(\mathcal{G})$$
$$\nabla \longmapsto F_\nabla - 2\pi i \frac{d}{n} \mathbf{1}_{\mathbb{E}} \omega_X .$$

- Symplectic quotient: $\mathcal{A}/\!\!/\mathcal{G} = \mu^{-1}(0)/\mathcal{G}$.

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The gauge-theoretic point of view

Theorem (Donaldson's version of Narasimhan–Seshadri)

An irreducible h-unitary connection ∇ satisfies

$$F_\nabla = -2\pi i \frac{d}{n} \mathbf{1}_{\mathbb{E}} \omega_X$$

if and only if $(\mathbb{E}, \nabla^{0,1})$ is stable.

- Consequence:

$$\mathcal{N}(n,d) \longleftrightarrow \mathcal{A} /\!\!/ \mathcal{G}.$$

The Hitchin system

- $T^*_{[E]}\mathcal{N}(n,d) \cong H^0(X, \mathrm{End}E \otimes K_X)$.
- $(E,\varphi) \in T^*\mathcal{N}(n,d)$, $\varphi : E \to E \otimes K_X$, characteristic polynomial:

$$P_\varphi(T) = \det(T - \varphi) = T^n + \sum_{i=1}^{n} \sigma_i(\varphi) T^{n-i}.$$

- **Hitchin fibration**:

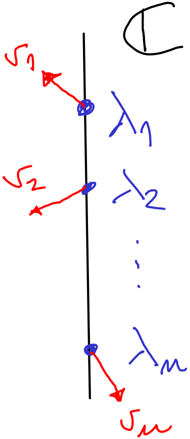
$$H : T^*\mathcal{N}(n,d) \longrightarrow \mathcal{B} := \bigoplus_{i=1}^{n} H^0(X, K_X^i)$$
$$(E,\varphi) \longmapsto (\sigma_1(\varphi), \dots, \sigma_n(\varphi))$$

- Fibres?

The spectral correspondence

$\varphi : \mathbb{C}^n \to \mathbb{C}^n$ endomorphism :

- $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ eigenvalues of φ,
- $v_1, \ldots, v_n \in \mathbb{C}^n$ eigenvectors of φ.



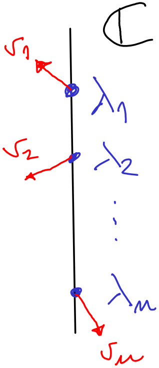
The spectral correspondence

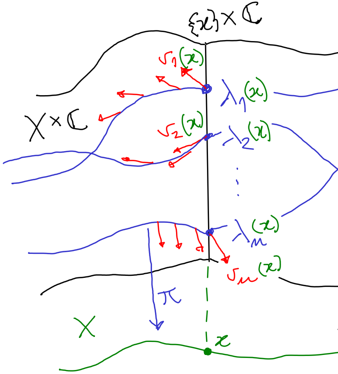
$E \to X$ holomorphic vector bundle.
$\varphi : E \to E$ vector bundle endomorphism :

- $\lambda_1(x), \dots, \lambda_n(x) \in \mathbb{C}$ eigenvalues of $\varphi_x : E_x \to E_x$,
- $v_1(x), \dots, v_n(x) \in \mathbb{C}^n$ eigenvectors of $\varphi_x : E_x \to E_x$.
- **Spectral curve**:

$$S_\varphi = \{(x, \lambda_i(x)), i = 1, \dots, n\} \subset X \times$$

- $S_\varphi \xrightarrow{\pi} X$ ramified cover.





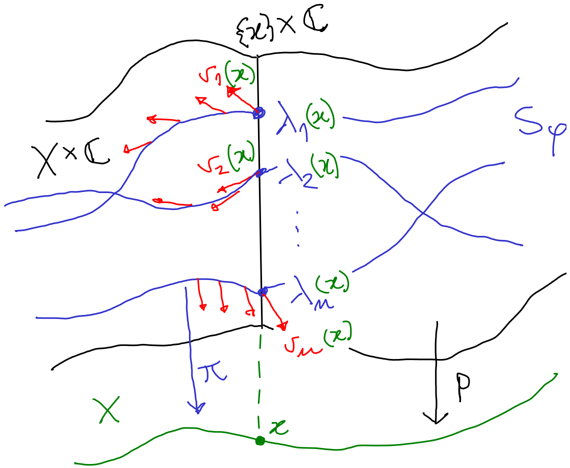
The spectral correspondence

$E \to X$ holomorphic vector bundle.
$\varphi : E \to E \otimes L$ **twisted** vector bundle endomorphism :

- $\lambda_1(x), \ldots, \lambda_n(x) \in L_x$ eigenvalues of $\varphi_x : E_x \to E_x \otimes L_x$,
- $v_1(x), \ldots, v_n(x) \in E_x$ eigenvectors of $\varphi_x : E_x \to E_x \otimes L_x$.
- **Spectral curve** (locally):

$$S_\varphi = \{(x, \lambda_i(x)), i = 1, \ldots, n\} \subset L.$$

- $S_\varphi \xrightarrow{\pi} X$ ramified cover.



The spectral correspondence

Theorem

Let $b \in \bigoplus_{i=1}^{n} H^0(X, L^i)$ *such that the spectral curve* S_b *is irreducible and smooth. There is a bijective correspondence*

$$\begin{aligned}\mathrm{Pic}^{\delta}(S_b) &\longleftrightarrow \{[(E,\varphi)] | P_{\varphi}(T) = P_b(T), \mathrm{rk} E = n, \deg E = d\} \\ [L] &\longmapsto [(\pi_* L, \pi_* \lambda)].\end{aligned}$$

with

$$\delta = d + \frac{n(n-1)}{2} d.$$

- Hitchin (1987) for $L = K_X$.
- Beauville, Narasimhan and Ramanan (1989) for general L.

The Hitchin system

- $(E, \varphi) \in T^*\mathcal{N}(n,d)$, characteristic polynomial:

$$P_\varphi(T) = \det(T - \varphi) = T^n + \sum_{i=1}^{n} \sigma_i(\varphi) T^{n-i}.$$

- **Hitchin fibration**:

$$H : T^*\mathcal{N}(n,d) \longrightarrow \mathcal{B} := \bigoplus_{i=1}^{n} H^0(X, K_X^i)$$
$$(E, \varphi) \longmapsto (\sigma_1(\varphi), \ldots, \sigma_n(\varphi))$$

- The fibres are open (stability) subsets of tori, $H^{-1}(b) \subset \mathrm{Pic}^\delta(S_b) \cong J(S_b)$.
- What about the remaining points of the Jacobian?
  Wider notion of stability for pairs (E, φ).

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Higgs bundles

Definition

A **Higgs bundle** is a pair (E, φ) where $E \to X$ is a holomorphic vector bundle and the **Higgs field** $\varphi : E \to E \otimes K_X$ is a K_X-twisted endomorphism.

A Higgs bundle (E, φ) is **stable** if for every proper holomorphic subbundle $E' \subset E$ such that $\varphi(E') \subset E' \otimes K_X$

$$\frac{\deg E'}{\mathrm{rk} E'} < \frac{\deg E}{\mathrm{rk} E}.$$

- **Moduli space of stable Higgs bundles**: $\mathcal{M}(n, d)$
- Hitchin (1987), Simpson (1988), Nitsure (1991)
- The Hitchin fibration extends to a proper map $H : \mathcal{M}(n, d) \to \mathcal{B}$.

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The gauge-theoretic point of view

$(\mathbb{E}, h)$ smooth complex Hermitian vector bundle of rank n and degree d over X.

- **Hitchin's equations**: $(\nabla, \varphi) \in \mathcal{A} \times \Omega^{1,0}(X, \mathrm{End}\ \mathbb{E})$.

$$\begin{cases} F_\nabla + [\varphi, \varphi^\dagger] = -2\pi i \frac{d}{n} \mathbf{1}_{\mathbb{E}} \omega_X \\ \nabla^{0,1}\varphi = 0. \end{cases}$$

- **Hitchin–Kobayashi correspondence**:

Irreducible solutions of Hitchin's equations $\longleftrightarrow$ Stable Higgs bundles

Higgs bundles twisted by a vector bundle

Definition

Let $V \to X$ be a rank r holomorphic vector bundle. A V-**twisted Higgs bundle** is a pair (E, φ), where $E \to X$ is a holomorphic vector bundle and $\varphi : E \to E \otimes V$ with

$$\varphi \wedge \varphi = 0 \text{ in } \mathrm{End}E \otimes \Lambda^2 V.$$

- The condition $\varphi \wedge \varphi = 0$ means that the components of φ commute, $[\varphi_i, \varphi_j] = 0$.
- Same stability condition than that for (classical) Higgs bundles.
- Moduli space: $\mathcal{M}_V(n, d)$.
- Studied by D. Xie and K. Yonekura in the context of $\mathcal{N} = 1$ supersymmetric gauge theories.

The gauge-theoretic point of view

$(\mathbb{E}, h)$ smooth complex Hermitian vector bundle of rank n and degree d over X.
Fix h_V Hermitian metric on the holomorphic vector bundle $V \to X$.

- **Generalized Hitchin's equations**: $(\nabla, \varphi) \in \mathcal{A} \times \Omega^0(X, \mathrm{End}\ \mathbb{E} \otimes V)$

$$\begin{cases} F_\nabla + [\varphi, \varphi^\dagger]_{h,h_V} \omega_X = -2\pi i \frac{d}{n} \mathbf{1}_{\mathbb{E}} \omega_X \\ (\nabla^{0,1} \otimes \mathbf{1}_V + \mathbf{1}_{\mathrm{End}\ \mathbb{E}} \otimes \nabla_V^{0,1})\varphi = 0 \\ \varphi \wedge \varphi = 0. \end{cases}$$

- **Hitchin–Kobayashi correspondence**:

  Irreducible solutions of generalized Hitchin's equations $\longleftrightarrow$ Stable V-twisted Higgs bundles

The generalized Hitchin system

- $(E, \varphi) \in \mathcal{M}_V(n, d)$, characteristic polynomial:

$$P_\varphi(T) = \det(T - \varphi) = T^n + \sum_{i=1}^{n} \sigma_i(\varphi) T^{n-i}.$$

- **Hitchin fibration**:

$$H_V : \mathcal{M}_V(n,d) \longrightarrow \bigoplus_{i=1}^{n} H^0(X, \mathrm{Sym}^i V)$$
$$(E, \varphi) \longmapsto (\sigma_1(\varphi), \ldots, \sigma_n(\varphi))$$

- Not surjective!! $\mathcal{B}_V = \mathrm{im} H_V$

The generalized spectral correspondence

- The spectral curve can be defined analogously in V, as the solutions of $P_\varphi(T)$.
- The number of equations is bigger than the dimension. However...
- **Main idea**: *Since the components of φ commute, generically they can be diagonalized simoultaneously and can be written as polynomials of one of them.*

Theorem

Take $b \in \mathcal{B}_V$ such that the spectral curve S_b is integral and smooth. Then there is a bijective correspondence

$$\mathrm{Pic}(S_b) \longleftrightarrow \{[(E,\varphi)] | S_\varphi = S_b\}\,.$$

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Open questions and further directions

Open questions:

- Complete the Hitchin-Kobayashi correspondence.
- GIT construction of the moduli space.
- Computation of the dimension of $\mathcal{M}_V(n,d)$ using deformation theory.
- Genus of the spectral curve and relationship between the degrees.
- Conditions for the curve to be smooth.
- Study the ramification divisor.

Further directions:

- Study the case $V = L_1 \oplus L_2$ with $L_1 \otimes L_2 = K_X$ (V is Calabi–Yau).
- Donaldson–Thomas invariants.

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