Multiplicative Higgs bundles

Guillermo Gallego

Freie Universität Berlin

Modern musings on monopoles
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Mini-complex 3-folds

A $\mathbb{C}$-valued function f on $\mathbb{R}_t \times \mathbb{C}_z$ is *mini-holomorphic* if $\partial_t f = 0$ and $\partial_{\bar{z}} f = 0$.

A *mini-complex 3-manifold* is a topological space Y locally modelled on $\mathbb{R} \times \mathbb{C}$ with mini-holomorphic coordinate changes.

Examples:

- Products: $\mathbb{R} \times C$, $[0,1] \times C$ and $S^1 \times C$, where C is a Riemann surface.
- Mapping tori: $S^1 \times_\tau C := [0,1] \times C/(0,p) \sim (1, \tau(p))$, where C is a Riemann surface and τ is an automorphism of C.

In the cases above, we put $\Omega^{0,1}(Y) = C^\infty(Y, \mathbb{C})dt \oplus \mathrm{pr}_X^* \Omega^{0,1}(C)$ and

$$\bar{\partial}_Y(f) = \partial_t(f) + \bar{\partial}_C(f).$$

Mini-holomorphic bundles

Let Y be a mini-complex 3-manifold and let E be a complex vector bundle of finite rank on Y. A *mini-holomorphic structure* on E is a differential operator

$$\bar{\partial}_{\mathcal{E}} : C^{\infty}(Y, E) \to C^{\infty}(Y, E \otimes \Omega_Y^{0,1})$$

satisfying the Leibniz rule

$$\bar{\partial}_{\mathcal{E}}(fs) = \bar{\partial}_Y(f)s + f\bar{\partial}_{\mathcal{E}}(s)$$

and the integrability condition

$$\bar{\partial}_{\mathcal{E}}^2 = 0.$$

Scattering and difference connections

Let $Y = S^1 \times_\tau C$ and let $\mathcal{E} = (E, \bar{\partial}_\mathcal{E})$ be a mini-holomorphic bundle on E.

The operator $\bar{\partial}_\mathcal{E}$ splits as $\bar{\partial}_{\mathcal{E},C} + \bar{\partial}_{\mathcal{E},S^1}$.

Consider the restriction $\mathcal{E}_0 = \mathcal{E}|_{0\times C}$, endowed with the operator $\bar{\partial}_{\mathcal{E},C}$. This determines a holomorphic vector bundle $\mathbf{E}$ on C.

The holonomy of the operator $\bar{\partial}_{\mathcal{E},S^1}$ along the circle S^1 determines an isomorphism

$$\varphi : \mathbf{E} \to \tau^*\mathbf{E}.$$

The integrability condition implies that φ is holomorphic.

The pair $(\mathbf{E}, \varphi)$ is called a τ*-difference connection*. If τ is the identity automorphism, then we say that $(\mathbf{E}, \varphi)$ is a *multiplicative Higgs bundle*.

Hermitian-Einstein monopoles

Let $Y = S^1 \times_\tau C$. Fix a Kähler metric g_C on C and put $g_Y = dt^2 + g_C$.

Let (E, H) be a Hermitian vector bundle on Y.

A *Hermitian-Einstein monopole* is a pair (A, Φ) formed by a H-unitary connection on E and by a section $\Phi \in \Omega^0(\mathrm{ad}_H(E))$ satisfying the *Hermite-Bogomolny equation*

$$F_A - *d_A\Phi = i\lambda\omega_C \mathrm{id}_E,$$

for some $\lambda \in \mathbb{R}$.

We can decompose this equation into two equations (real and complex)

$$\begin{aligned} F_{A,C} - \partial_{A,t}\Phi &= i\lambda \mathrm{id}_E \\ [\bar{\partial}_{A,C}, \partial_{A,t} - i\Phi] &= 0. \end{aligned}$$

Mini-holomorphic bundle induced from a monopole

With a Hermitian-Einstein monopole (A, Φ) we can associate the operator

$$\bar{\partial}_{(A,\Phi)} = \bar{\partial}_{A,C} + \partial_{A,t} - i\Phi,$$

which satisfies the Leibniz rule $\bar{\partial}_{(A,\Phi)}(fs) = \bar{\partial}_Y(f)s + f\bar{\partial}_{(A,\Phi)}(s)$.

The equation $[\bar{\partial}_{A,C}, \partial_{A,t} - i\Phi] = 0$ implies the integrability condition $\bar{\partial}^2_{(A,\Phi)} = 0$.

Therefore, (A, Φ) determines a mini-holomorphic structure on E.

3-dimensional Chern correspondence

Consider a mini-holomorphic bundle $\mathcal{E} = (E, \bar{\partial}_{\mathcal{E}})$. Associated with any Hermitian metric H on $\mathcal{E}$ there is a unique pair (A_H, Φ_H) such that $\bar{\partial}_{\mathcal{E}} = \bar{\partial}_{(A_H, \Phi_H)}$.

We call this (A_H, Φ_H) the *Chern pair*.

We say that a Hermitian metric H on $\mathcal{E}$ is Hermitian-Einstein-Bogomolny if the Chern pair (A_H, Φ_H) is a Hermitian-Einstein monopole.

Hitchin-Kobayashi corrrespondence

A mini-holomorphic bundle $\mathcal{E}$ is *stable* if, for every mini-holomorphic subbundle $\mathcal{F} \subset \mathcal{E}$ we have the following inequality of slopes

$$\deg(\mathcal{F})/\mathrm{rk}(\mathcal{F}) < \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E}).$$

We say that $\mathcal{E}$ is *polystable* if it is a sum of stable bundles of the same slope.

Theorem (Charbonneau-Hurtubise, Mochizuki, Yoshino...)

A mini-holomorphic bundle is stable if and only if it admits a Hermitian-Einstein-Bogomolny metric.

B. Smith gave a version for principal bundles.

IDEA: Monopoles are 3D reductions of instantons. Solve HK in $Y \times S^1$ and reduce. Resolve Dirac-type singularities locally through the Hopf fibration (Kronheimer's Master thesis).

Dirac-type singularities

In general we want to work on open mini-complex 3-manifolds, and thus we need to impose *boundary conditions*.

In this talk, we consider 3-manifolds Y of the form $Y = \bar{Y} \setminus Z$, where $\bar{Y}$ is a compact mini-complex 3-manifold and $Z \subset \bar{Y}$ is a finite subset.

We say that a HE monopole (A, Φ) of rank r has a *Dirac-type singularity* of charge $\mathbf{k} = (k^1, \ldots, k^r)$ at a point $y \in Y$ if

- there exists a small ball B around y such that $(E, H)|_{B \setminus \{y\}}$ is decomposed into a sum of Hermitian line bundles L_i of degrees $\deg L_i = \int_{\partial B} c_1(L_i) = k^i$,
- in the above decomposition, we have the estimates

$$\begin{cases} \Phi = \frac{i}{2R} \sum_{i=1}^{r} k^i \mathrm{id}_{L_i} + O(1) \\ d_A(R\Phi) = O(1), \end{cases}$$

  where R is the radial coordinate in B.

Meromorphic scattering

Suppose now that $\bar{Y} = S^1 \times_\tau C$ is a mapping torus and $Z = \{y_1 = (t_1, p_1), \dots, y_n = (t_n, p_n)\}$, and consider a mini-holomorphic bundle $\mathcal{E}$ on $Y = \bar{Y} \setminus Z$ induced by a pair (A, Φ) with Dirac-type singularities of charge $\mathbf{k}_i$ at the points y_i.

This $\mathcal{E}$ determines a holomorphic vector bundle $\mathbf{E}$ on C, and an isomorphism $\varphi : \mathbf{E} \to \tau^*\mathbf{E}$ defined only away from the points $p_1, \dots, p_n \in C$.

It is not hard to check that (A, Φ) has a Dirac-type singularity of charge $\mathbf{k}_i$ at the point y_i if and only if, in a small disk near p_i, the map φ has the form

$$\varphi(z) = g_1(z)\mathrm{diag}(z^{k_i^1}, \dots, z^{k_i^r})g_2(z),$$

with $g_1, g_2 \in \mathrm{GL}_r(\mathbb{C}[[z]])$.

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Nonabelian Hodge theory

Let C be a compact Riemann surface and fix a Kähler metric g_C on it. Fix a rank r Hermitian vector bundle (E, H) on C.

$$\mathcal{M} = \{\text{Solutions to Hitchin's equations on } E\} / \text{gauge}$$
$$J_\lambda = \{\text{Twistor family of complex structures on } \mathcal{M}\} .$$

$(\mathcal{M}, J_\lambda)$ is the moduli space of pairs $(\mathbf{E}, \nabla_\lambda)$ formed by a holomorphic vector bundle $\mathbf{E}$ and a λ-connection ∇_λ on $\mathbf{E}$.

At $\lambda = 0$ we obtain the *Dolbeault moduli space* parametrizing Higgs bundles.

At $\lambda = 1$ we obtain the *de Rham moduli space* parametrizing bundles with holomorphic connection.

“Multiplicative nonabelian Hodge theory”

Let (C, g_C) be a Calabi-Yau Riemann surface (or simply $C = \mathbb{C}, \mathbb{C}^*$ or an elliptic curve) and fix a rank r Hermitian vector bundle (E, H) on $C \times S^1$.

$$\mathcal{M} = \{\text{Solutions to Hermite-Bogomolny equations on } E\} / \text{gauge}$$

$$J_\lambda = \{\text{Twistor family of complex structures on } \mathcal{M}\}.$$

$(\mathcal{M}, J_\lambda)$ is the moduli space of pairs $(\mathbf{E}, \varphi)$ formed by a holomorphic vector bundle $\mathbf{E}$ and a τ_λ-difference connection, for τ_λ some automorphism of C determined by λ. If λ_0 is such that $\tau_{\lambda_0} = \text{id}$, then $(\mathcal{M}, J_{\lambda_0})$ is the moduli space of multiplicative Higgs bundles on C.

Periodic monopoles à la Mochizuki

- **Rational case**. $C = \mathbb{C}$. $\mathcal{M}$ parametrizes periodic monopoles on $\mathbb{R}^3$ (Cherkis-Kapustin type asymptotics). $\tau_\lambda(z) = z + \lambda$. Bundles on C with a τ_λ-difference connection are rational λ-difference modules. $\lambda = 0$: multiplicative Higgs bundles on $\mathbb{C}$.
- **Trigonometric case**. $C = \mathbb{C}^*$. $\mathcal{M}$ parametrizes doubly periodic monopoles on $\mathbb{R}^3$ (Cherkis-Ward type asymptotics). $\tau_\lambda(z) = q_\lambda z$, for $q_\lambda = \exp\left(-2\pi \frac{1-\lambda^2}{1+\lambda^2}\right)$. Bundles on C with a τ_λ-difference connection are trigonometric q_λ-difference modules. $\lambda = 1$: multiplicative Higgs bundles on $\mathbb{C}^*$.
- **Elliptic case**. $C = \mathbb{C}/\Lambda$ is an elliptic curve. $\mathcal{M}$ parametrizes triply periodic monopoles on $\mathbb{R}^3$. $\tau_\lambda(z + \Lambda) = (z + \lambda) + \Lambda$. Bundles on C with a τ_λ-difference connection are elliptic λ-difference modules. $\lambda = 0$: multiplicative Higgs bundles on C.

The Betti side

The Riemann-Hilbert correspondence associates with a holomorphic bundle with holomorphic connection on a Riemann surface C a representation of the fundamental group of C.

Analogues of the Riemann-Hilbert correspondence for difference modules have been a topic of study for a long time.

Kontsevich–Soibelman: Generalized Riemann-Hilbert correspondence as a derived equivalence of A_∞ categories. Betti side is Fukaya category of Lagrangian submanifolds of some holomorphic symplectic manifold.

A more concrete example: A q-difference module determines a vector bundle on the elliptic curve $\mathbb{E}_q = \mathbb{C}^*/q^{\mathbb{Z}}$ (quotienting by τ_q). The asymptotics at 0 and ∞ determines anti-Harder-Narasimhan filtrations on the corresponding bundle on $\mathbb{E}_q$.

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The Hitchin fibration

C compact Riemann surface. G complex reductive group of rank r.

Moduli stack of *Higgs bundles*:

$$\mathcal{M}_G = \left\langle (\mathbf{E}, \varphi) : \mathbf{E} \to C \text{ hol. } G\text{-bundle}, \varphi \in H^0(C, \mathrm{ad}(\mathbf{E}) \otimes K_C) \right\rangle .$$

Invariant polynomials: $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[a_1, \ldots, a_r]$, $d_i = \deg(a_i)$.

Hitchin base: $\mathcal{A}_G = H^0(C, K_C \times^{\mathbb{C}^*} (\mathfrak{g} /\!\!/ G)) = \bigoplus_{i=1}^r H^0(C, K_C^{d_i})$.

Hitchin map:

$$h_G : \mathcal{M}_G \to \mathcal{A}_G,\ (\mathbf{E}, \varphi) \mapsto (a_1(\varphi), \ldots, a_r(\varphi)).$$

Global version of the natural map $[\mathfrak{g}/G] \to \mathfrak{g} /\!\!/ G = \mathfrak{t}/W$.

For classical groups it is just the characteristic polynomial.

Complete integrability (Hitchin, Donagi-Gaitsgory)

For general values of $a \in \mathcal{A}_G$, the fibre $h_G^{-1}(a)$ is isomorphic to a Beilinson 1-motive of the form

$$\mathcal{P}_{G,a} = BZ_G \times P_{\tilde{C}_a,G} \times \pi_1(G).$$

$P_{\tilde{C}_a,G}$ is a "generalized Prym variety", an abelian variety associated with the *cameral cover* $\tilde{C}_a \to C$, obtained as a pullback from $K_C \times^{\mathbb{C}^*} \mathfrak{t} \to K_C \times^{\mathbb{C}^*} (\mathfrak{t}/W)$.

For classical groups, there is an equivalent description in terms of the spectral cover $\hat{C}_a \to C$ parametrizing the eigenvalues of the Higgs field.

For example, for $G = \mathsf{SL}_n(\mathbb{C})$,

$$\mathcal{P}_{\mathsf{SL}_n(\mathbb{C})} = B\mu_n \times \mathrm{Prym}(\hat{C}_a/C).$$

Langlands duality and SYZ mirror symmetry

Let G and $G^\vee$ be two semisimple Langlands dual groups. For $a \in \mathcal{A}_{G^\vee}$, the fibre $h_{G^\vee}^{-1}(a)$ is isomorphic to

$$\mathcal{P}_{\tilde{C}_a, G^\vee} = BZ_{G^\vee} \times P_{\tilde{C}_a, G^\vee} \times \pi_1(G^\vee) = B(\pi_1(G)^*) \times P_{\tilde{C}_a, G^\vee} \times Z_G^*,$$

where $(-)^*$ stands for Cartier duality.

The Killing form $\mathfrak{t} \to \mathfrak{t}^*$ is a W-equivariant isomorphism, so it induces an isomorphism of the Hitchin bases $\mathcal{A}_G \to \mathcal{A}_{G^\vee}$, lifting to an isomorphism of the cameral covers. Donagi and Pantev proved that the abelian varieties $P_{\tilde{C}_a, G}$ and $P_{\tilde{C}_a, G^\vee}$ are dual to each other.

For example, Hausel and Thaddeus had already proven that

$$\mathcal{P}_{\mathsf{PGL}_n(\mathbb{C})} = \mathsf{Prym}^\vee(\hat{C}_a/C) \times \mathbb{Z}/n\mathbb{Z} = \mathsf{Hom}(\mathcal{P}_{\mathsf{SL}_n(\mathbb{C})}, B\mathbb{C}^*),$$

where $\mathsf{Prym}^\vee(\hat{C}_a/C) \cong \mathsf{Prym}(\hat{C}_a/C)/\pi^*\mathsf{Jac}(C)[2]$ is the dual of the Prym variety of $\pi : \hat{C}_a \to C$.

The multiplicative Hitchin fibration

C compact Riemann surface. G complex semisimple group of rank r.
Fix $T \subset B \subset G$. This gives simple roots $\{\alpha_1, \ldots, \alpha_r\}$ and fundamental weights $\{\omega_1, \ldots, \omega_r\}$.
$p_1, \ldots, p_n \in C$, $\lambda_1, \ldots, \lambda_n \in \Lambda_T^+$, $D = \sum_{i=1}^n \lambda_i p_i$.

Moduli stack of multiplicative Higgs bundles with pole data given by D:
$\mathcal{M}_{G,D} =$
$\left\langle (\mathbf{E}, \varphi) : \mathbf{E} \to C \text{ } G\text{-bundle}, \varphi \in \Gamma(C \setminus |D|, \mathbf{E} \times^{G,\mathrm{Ad}} G^{\mathrm{sc}}), \text{ pole}(\varphi) = D \right\rangle$.

Multiplicative Hitchin base:
$\mathcal{A}_{G,D} = \bigoplus_{i=1}^r H^0(C, \mathcal{O}_C(\langle \omega_i, D \rangle)) = \bigoplus_{i=1}^r H^0(C, \mathcal{O}_C(\sum_{j=1}^n \langle \omega_i, \lambda_j \rangle p_j))$.

Multiplicative Hitchin map:
$h_{G,D} : \mathcal{M}_{G,D} \to \mathcal{A}_{G,D} : (\mathbf{E}, \varphi) \mapsto (\mathrm{tr}(\rho_{\omega_1}(\varphi)), \ldots, \mathrm{tr}(\rho_{\omega_r}(\varphi)))$.

Introduced by Hurtubise–Markman (for C = elliptic curve – so triply-periodic monopoles). Works for any C, but lose HK structure.

The monoid POV (Frenkel–Ngô, Bouthier, Chi, Wang)

(also Hurtubise–Markman)

Want: "stacky description" similar to $[K_C \times^{\mathbb{C}^*} \mathfrak{g}/G] \to (K_C \times^{\mathbb{C}^*} \mathfrak{g}) /\!\!/ G$.
Have: "meromorphic" $[G^{\mathrm{sc}}/G] \to G^{\mathrm{sc}} /\!\!/ G$.

IDEA: "Partially compactify" $G^{\mathrm{sc}}_C \to C$ to some $G^{\mathrm{sc}}_D \to C$, and consider $[G^{\mathrm{sc}}_D/G] \to G^{\mathrm{sc}}_D /\!\!/ G$.

- Take the Vinberg monoid $\mathrm{Env}(G^{\mathrm{sc}}) \to \mathfrak{t}$. Flat family of copies of G^{sc} degenerating at the hyperplanes $\alpha_i = 0$.
- Construct the T-torsor $\mathscr{L}_D = \mathscr{O}_C(D)$ and the vector bundle $\mathscr{L}_D \times^T \mathfrak{t} = \bigoplus_{i=1}^r \mathscr{O}_C(\sum_{j=1}^n \langle \alpha_i, \lambda_j \rangle p_j)$, which admits the section $1_D = (1, \dots, 1)$ (since the λ_i are dominant).
- G^{sc}_D is obtained as the pullback of $\mathscr{L}_D \times^T \mathrm{Env}(G^{\mathrm{sc}})$ through the section 1_D.

Bundle chains (j.w. Hurtubise and García-Prada)

Martens–Thaddeus described reductive monoids in terms of moduli spaces of "bundle chains". This provides a different description of multiplicative Higgs bundles. We consider tuples $(S, E_0, E, \Psi_\pm)$ as follows:

- $S \to C$ is a flat family of $\mathbb{CP}^1$'s degenerating to a wedge of $\mathbb{CP}^1$'s over each singularity $p_i \in C$. It comes with sections $p_\pm : C \to S$ (the north and south poles) and with exceptional divisors $D_i \subset S$, corresponding to the "south $\mathbb{CP}^1$" over each singularity. The dominant cocharacters λ_i determine a G-bundle $\mathcal{E}_D = \mathcal{O}_S(\sum_i \lambda_i D_i) \to S$. This bundle has local framings, near a p_i, of the form 1 at $p_+(C)$ and z^{λ_i} at $p_-(C)$, away from p_i.
- $E_0 \to C$ is a G-bundle and $E \to S$ is a $\mathbb{C}^*$-equivariant G-bundle.
- $\Psi_\pm : E|_{p_\pm(C)} \to p_{\pm,*}E_0$ isomorphisms such that, locally, near a nodal fibre, a trivialization of E_0 induces an isomorphism of E with $\mathcal{E}_D$ and identifies Ψ_+ with 1 and Ψ_- with z^{λ_i}.

Complete integrability

Let $T^{\mathrm{sc}}_D \to C$ be the closure of T^{sc}_C inside G^{sc}_D and note that $G^{\mathrm{sc}}_D /\!\!/ G = T^{\mathrm{sc}}_D/W$ and $\mathcal{A}_{G,D} = H^0(C, T^{\mathrm{sc}}_D/W)$.

Given $a \in \mathcal{A}_{G,D}$, we construct the corresponding cameral cover $\tilde{C}_a \to C$ as the pullback through a of the cover

$$T^{\mathrm{sc}}_D \to T^{\mathrm{sc}}_D/W.$$

Assumming "ampleness" on D, for a generic a, the cameral curve $\tilde{C}_a$ is smooth and the cover has simple Galois ramification in the sense of Donagi–Pantev. For these a, the multiplicative Hitchin fibre is again isomorphic to a Beilinson 1-motive, of the form

$$\mathcal{P}_{G,a} = BZ_G \times P_{\tilde{C}_a,G} \times \pi_1(G).$$

Highlights: G. Wang's proof of the Fundamental Lemma for the spherical Hecke algebras.

SYZ mirror symmetry. Simply-laced case

If G is simply laced then $G^{\mathrm{sc}} \cong (G^\vee)^{\mathrm{sc}}$. In that case the Hitchin bases and the cameral covers for G and $G^\vee$ are identified, and we have

$$\mathcal{P}_{G^\vee,a} = BZ_{G^\vee} \times P_{\tilde{C}_a,G^\vee} \times \pi_1(G^\vee) = \mathrm{Hom}(\mathcal{P}_{G,a}, B\mathbb{C}^*),$$

where we are using the duality of $P_{\tilde{C}_a,G}$ and $P_{\tilde{C}_a,G^\vee}$ proved by Donagi–Pantev.

What if G is not simply-laced? The iso. $\mathfrak{t} \to \mathfrak{t}^*$ provided by the Killing form is not enough in the multiplicative case. We obtain a new kind of duality.

Twisted multiplicative Higgs bundles

G semisimple, simply-laced, simply-connected complex group.

$\theta \in \mathrm{Aut}(G)$ diagram automorphism. Consider twisted conjugation

$$g * h = gh\theta(g)^{-1}.$$

Multiplicative Hitchin fibrations are constructed from $[G/_\theta G] \to G /\!\!/_\theta G$.

Twisted multiplicative Higgs bundles:
(E, φ), $E \to C$ G-bundle, $\varphi : E \to \theta(E)$ meromorphic iso.

Arise as fixed points in the space of monopoles, after taking "half-way" scattering, or simply as "twisted periodic" monopoles.

The key to duality: $G /\!\!/_\theta G \cong G_\theta /\!\!/ G_\theta$ (Mohrdieck), where $G_\theta = ((G^\vee)^{\theta^\vee})^\vee$ is the "coinvariant group".

SYZ mirror symmetry. Twisted case

Let (G, θ) as before, and put $H = G^\theta$. Suppose that G does not have a component isomorphic to $\mathrm{SL}_{2\ell+1}$.

Consider a twisted multiplicative Hitchin fibration $h_{G,\theta} : \mathcal{M}_{G,\theta} \to \mathcal{A}_{G,\theta}$ associated with (G, θ) and a multiplicative Hitchin fibration $h_{H^\vee} : \mathcal{M}_{H^\vee} \to \mathcal{A}_{H^\vee}$ associated with $H^\vee$ (note that $(H^\vee)^{\mathrm{sc}} = G_\theta$).

The bases $\mathcal{A}_{G,\theta}$ and $\mathcal{A}_{H^\vee}$ are identified, through the isomorphism $G /\!\!/_\theta G \cong G_\theta /\!\!/ G_\theta$.

For generic a, the fibres $h_{G,\theta}^{-1}(a)$ and $h_{H^\vee}^{-1}(a)$ are dual Beilinson 1-motives.

In the case that $G = \mathrm{SL}_{2\ell+1}$, one matches two different twisted multiplicative Hitchin fibrations.

Summary of duality

Pair	Type of J	Type of cameral cover	Type of dual J	Dual pair
$(\mathrm{SL}_r, \mathrm{SL}_r)$	SL_r	A_r	PGL_r	$(\mathrm{PGL}_r, \mathrm{SL}_r)$
$(\mathrm{SO}_{2r}, \mathrm{Spin}_{2r})$	SO_{2r}	D_r	SO_{2r}	$(\mathrm{SO}_{2r}, \mathrm{Spin}_{2r})$
$(\mathrm{Spin}_{2r}, \mathrm{Spin}_{2r})$	Spin_{2r}	D_r	PSO_{2r}	$(\mathrm{PSO}_{2r}, \mathrm{Spin}_{2r})$
$(\mathsf{E}_6, \mathsf{E}_6)$	E_6	E_6	E_6	$(\mathsf{E}_6, \mathsf{E}_6)$
$(\mathsf{E}_7, \mathsf{E}_7)$	E_7	E_7	E_7	$(\mathsf{E}_7, \mathsf{E}_7)$
$(\mathsf{E}_8, \mathsf{E}_8)$	E_8	E_8	E_8	$(\mathsf{E}_8, \mathsf{E}_8)$
$(\mathrm{SL}_{2\ell}, \mathrm{SL}_{2\ell}\,\theta)$	$\mathrm{Sp}_{2\ell}$	$\mathsf{C}_\ell \sim \mathsf{B}_\ell$	$\mathrm{SO}_{2\ell+1}$	$(\mathrm{SO}_{2\ell+1}, \mathrm{Spin}_{2\ell+1})$
$(\mathrm{Spin}_{2\ell+2}, \mathrm{Spin}_{2\ell+2}\theta)$	$\mathrm{Spin}_{2\ell+1}$	$\mathsf{B}_\ell \sim \mathsf{C}_\ell$	$\mathrm{PSp}_{2\ell}$	$(\mathrm{PSp}_{2\ell}, \mathrm{Sp}_{2\ell})$
$(\mathsf{E}_6, \mathsf{E}_6\theta)$	F_4	F_4	F_4	$(\mathsf{F}_4, \mathsf{F}_4)$
$(\mathrm{Spin}_8, \mathrm{Spin}_8\theta_3)$	G_2	G_2	G_2	$(\mathsf{G}_2, \mathsf{G}_2)$
$(\mathrm{SL}_3, \mathrm{SL}_3\,\theta)$	$\mathrm{SO}_3\ (*)$	A_1	$\mathrm{Sp}_2\ (*)$	$(\mathrm{SL}_3, \mathrm{SL}_3\,\vartheta)$
$(\mathrm{SL}_{2\ell+1}, \mathrm{SL}_{2\ell+1}\,\theta)$	$\mathrm{SO}_{2\ell+1}\ (*)$	$\mathsf{B}_\ell \sim \mathsf{C}_\ell$	$\mathrm{Sp}_{2\ell}\ (*)$	$(\mathrm{SL}_{2\ell+1}, \mathrm{SL}_{2\ell+1}\,\vartheta)$

Summary of duality. Duality of affine Dynkin diagrams

Pair	Affine Dynkin diagram	Dual affine Dynkin diagram	Dual pair
$(\mathrm{SL}_r, \mathrm{SL}_r)$	$\mathsf{A}_r^{(1)}$	$\mathsf{A}_r^{(1)}$	$(\mathrm{PGL}_r, \mathrm{SL}_r)$
$(\mathrm{SO}_{2r}, \mathrm{Spin}_{2r})$	$\mathsf{D}_r^{(1)}$	$\mathsf{D}_r^{(1)}$	$(\mathrm{SO}_{2r}, \mathrm{Spin}_{2r})$
$(\mathrm{Spin}_{2r}, \mathrm{Spin}_{2r})$	$\mathsf{D}_r^{(1)}$	$\mathsf{D}_r^{(1)}$	$(\mathrm{PSO}_{2r}, \mathrm{Spin}_{2r})$
$(\mathsf{E}_6, \mathsf{E}_6)$	$\mathsf{E}_6^{(1)}$	$\mathsf{E}_6^{(1)}$	$(\mathsf{E}_6, \mathsf{E}_6)$
$(\mathsf{E}_7, \mathsf{E}_7)$	$\mathsf{E}_7^{(1)}$	$\mathsf{E}_7^{(1)}$	$(\mathsf{E}_7, \mathsf{E}_7)$
$(\mathsf{E}_8, \mathsf{E}_8)$	$\mathsf{E}_8^{(1)}$	$\mathsf{E}_8^{(1)}$	$(\mathsf{E}_8, \mathsf{E}_8)$
$(\mathrm{SL}_{2\ell}, \mathrm{SL}_{2\ell}\,\theta)$	$\mathsf{A}_{2\ell-1}^{(2)}$	$\mathsf{B}_\ell^{(1)}$	$(\mathrm{SO}_{2\ell+1}, \mathrm{Spin}_{2\ell+1})$
$(\mathrm{Spin}_{2\ell+2}, \mathrm{Spin}_{2\ell+2}\theta)$	$\mathsf{D}_{\ell+1}^{(2)}$	$\mathsf{C}_\ell^{(1)}$	$(\mathrm{PSp}_{2\ell}, \mathrm{Sp}_{2\ell})$
$(\mathsf{E}_6, \mathsf{E}_6\theta)$	$\mathsf{E}_6^{(2)}$	$\mathsf{F}_4^{(1)}$	$(\mathsf{F}_4, \mathsf{F}_4)$
$(\mathrm{Spin}_8, \mathrm{Spin}_8\theta_3)$	$\mathsf{D}_4^{(3)}$	$\mathsf{G}_2^{(1)}$	$(\mathsf{G}_2, \mathsf{G}_2)$
$(\mathrm{SL}_3, \mathrm{SL}_3\,\theta)$	$\mathsf{A}_2^{(2)}$	$\mathsf{A}_2^{(2)}$	$(\mathrm{SL}_3, \mathrm{SL}_3\,\vartheta)$
$(\mathrm{SL}_{2\ell+1}, \mathrm{SL}_{2\ell+1}\,\theta)$	$\mathsf{A}_{2\ell}^{(2)}$	$\mathsf{A}_{2\ell}^{(2)}$	$(\mathrm{SL}_{2\ell+1}, \mathrm{SL}_{2\ell+1}\,\vartheta)$

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Higgs bundles and real forms

Let $G_{\mathbb{R}} = G^{\sigma}$ be a real form of the complex group G, determined by a conjugation $\sigma \in \mathrm{Conj}(G)$. We also fix the maximal compact real form $K = G^{\sigma_K} \subset G$, determined by a conjugation $\sigma_K \in \mathrm{Conj}(G)$. The composition $\theta = \sigma \circ \sigma_K \in \mathrm{Aut}_2(G)$ is a holomorphic involution of G.

The involution θ induces a decomposition $\mathfrak{g} = \mathfrak{g}^{\theta} \oplus \mathfrak{m}$ in $+1$ and -1 eigenspaces. The fixed point subgroup G^{θ} acts on each of the pieces by restriction of the adjoint action of G on $\mathfrak{g}$.

$G_{\mathbb{R}}$-Higgs bundles:
(E, φ), $E \to C$ G^{θ}-bundle, $\varphi \in H^0(C, (E \times^{G^{\theta}} \mathfrak{m}) \otimes K_C)$.

$G_{\mathbb{R}}$-Higgs bundles yield representations $\pi_1(C) \to G_{\mathbb{R}}$ under the nonabelian Hodge correspondence.

Branes and relative Langlands duality

The moduli space of $G_{\mathbb{R}}$-Higgs bundles is a complex Lagrangian subvariety of the moduli space of G-Higgs bundles, with its natural complex symplectic structure. In terms of the "hyperkähler mirror symmetry" of Kapustin–Witten, it gives the support of a (BAA)-brane. This can be seen in two ways: (1) $G_{\mathbb{R}}$-Higgs bundles arise as fixed points of the involution $(E, \varphi) \mapsto (\theta(E), -\theta(\varphi))$. (2) The symmetric variety G/G^{θ} determines a "Gaiotto lagrangian" (Ginzburg–Rozenblyum).

Nadler proved a "real group version" of geometric Satake. The resulting Tannaka group is a complex subgroup $H(G_{\mathbb{R}}) \subset G^{\vee}$ which serves as "dual" for $G_{\mathbb{R}}$. The mirror (BBB)-brane is conjecturally supported on the moduli space of $H(G_{\mathbb{R}})$-Higgs bundles.

Some names: Gaiotto–Witten, Hausel–Hitchin, Biswas–García-Prada–Ramanan, Baraglia–Shaposnik, Hameister–Luo–Morrissey, Chen–Hsiao–Yang...

"$G_{\mathbb{R}}$" multiplicative Higgs bundles

(j.w. García-Prada and my thesis)

$G_{\mathbb{R}}$ is just a name, the "Betti side" is still unclear to me.

In the multiplicative situation, the vector space $\mathfrak{m}$ must be replaced by the whole symmetric variety $M = G/G^\theta$. The multiplicative version of $G_{\mathbb{R}}$-Higgs bundles is then given by pairs of the form (E, φ) with $E \to C$ a G^θ-bundle and $\varphi \in \Gamma(C, E \times^{G^\theta} (G/G^\theta))$ meromorphic.

Similar "monoid POV" by using Guay's enveloping embedding.

They appear as fixed points of $(E, \varphi) \mapsto (\theta(E), \theta(\varphi)^{-1})$, although the description of the fixed points is more complicated than in the Lie algebra case. When the moduli space is hyperkähler, they determine a (BAA)-brane. Mirror brane? Analogue of Gaiotto's Lagrangian?

Contents

Further directions and open problems

- Duality: Can we go beyond the generic locus? "Multiplicative geometric Langlands"?
- More duality and mirror symmetry: formulate "Gaiotto's Langrangian", generalize to spherical varieties, identify dual branes...
- "Multiplicative non-abelian Hodge theory". Give a complete statement of the correspondence, identify the "real group" part inside the "Betti side", higher Teichmüller components...Is there something for general curves?
  IDEA (WIP w. Hurtubise and García-Prada): Bogomolny equations = "loop group Hitchin equations" (similar to Garland's study of calorons)
- Gauge equations and GIT: Find "intrinsic" gauge equations (not going through $C \times S^1$) and an associated Hitchin–Kobayashi correspondence (à la Mundet i Riera). Provide an "algebraic" GIT construction of the moduli space. WIP w. Hurtubise and García-Prada

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🎉🎂🎊 Happy Birthday Bogomolny Equation! 🎊🎂🎉

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$$F_A = *d_A\Phi$$

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(For more information: `guillegallego.xyz`)