Vector bundles over algebraic curves

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Contents

Chapter 1

(Semi-)stable vector bundles

Introduction

Let X be an irreducible smooth projective curve over an algebraically closed commutative field k.

We say that a vector bundle E on X is semi-stable (resp. stable) if for every proper vector subbundle F of E we have

$$\deg(F)/\mathrm{rk}(F) \leq \deg(E)/\mathrm{rk}(E) \ \ (\text{resp.} <) \ .$$

(Semi-)stable bundles are interesting because of the fact that for given integers r, d, with $r \geq 2$, there exists an algebraic variety $U_s(r,d)$ where the set of closed points is the set $S'(r,d)$ of isomorphism classes of stable vector bundles of rank r and degree d. The closed points of a natural completion $U(r,d)$ of this variety can be seen as equivalence classes of semi-stable bundles. The varieties above are defined up to isomorphism by universal properties. If r and d are coprime, every semi-stable vector bundle is stable, and in that case there exists a "Poincaré bundle" over $U(r,d)\times X$, that is a vector bundle V such that for every closed point z of $U(r,d)$, the vector bundle V over X is in the isomorphism class z (an element of $S'(r,d)$).

Another justification of the above definitions: for every family $\mathscr{F}$ of vector bundles over X parametrized by a Noetherian k-scheme T, the set of points t of T such that $\mathscr{F}_t$ is (semi-)stable is open in T.

In order to define an algebraic variety where the set of closed points is $S'(r,d)$, we first construct a family $\mathscr{F}$ of vector bundles of rank r and degree d "containing" all the semi-stable bundles of rank r and degree d, parametrized by a Noetherian k-scheme R. For this we use the Grothendieck schemes (or "Quot schemes"). In fact, R is an open subset of

$$Q = \mathrm{Quot}^{P}_{\mathscr{O}\otimes k^p/X/k}$$

(P being the Hilbert polynomial of a rank r vector bundle of degree d and given the choice of an ample line bundle over X).

The reductive group $\mathrm{PGL}(p)$ acts on R, and the quotient $R/\mathrm{PGL}(p)$, as a set, is the set of isomorphism classes of vector bundles of the family $\mathscr{F}$.

We are thus reduced to "quotient" R by $\mathrm{PGL}(p)$. This can only be done on an open subset of R, formed by the so-called "semi-stable" points for the action of $\mathrm{PGL}(p)$. We can show that a point q of R is (semi-)stable if and only if $\mathscr{F}_q$ is, which justifies the definitions above.

Unfortunately the study of the action of $\mathrm{PGL}(p)$ on Q is difficult, and we are led to use a better known variety Y over which $\mathrm{SL}(p)$ acts, with an $\mathrm{SL}(p)$-morphism

$$\tau : R \longrightarrow Y .$$

We can then compare the points q of R such that $\mathscr{F}_q$ is (semi-)stable, and the (semi-)stable points of Y of the action of $\mathrm{SL}(p)$: one finds that the latter are the images by τ of the former, and that τ is injective. We can deduce the construction of the desired varieties. We call the variety $U(r,d)$ (resp. $U_s(r,d)$) the *moduli variety* of semi-stable (resp. stable) bundles of rank r and degree d.

On Section I, we give the main definitions and elementary properties of (semi-)stable vector bundles.

On Section II, we specify the required properties of a moduli variety of (semi-)stable vector bundles.

On Section III [REF], we carry out the construction of the moduli varieties. Results from Mumford's Theory are stated without proof.

On Section IV [REF], we treat the case where k is the field of complex numbers. We can then stablish a relation between semi-stable vector bundles over X and unitary representations of the fundamental group of X.

On section 5 [REF], we determine the singular points of the moduli varieties.

On section 6 [REF] we give some results without proof, concerning: the Picard variety of the moduli varieties, the existence of Poincaré bundles, the rationality of the moduli varieties and their topological properties in the case where k is the field of complex numbers.

I Stable bundles, semi-stable bundles. Some properties

Definition 1. A vector bundle E on X is *semi-stable* (resp. *stable*) if for every proper subbundle F of E, we have:

$$\mu(F) \leq \mu(E) \ \text{(resp. } \mu(F) < \mu(E)\text{)} \ .$$

Equivalent definitions: the bundle E is semi-stable (resp. stable) if and only if one of the following three properties is verified:

(i) For every proper quotient bundle F of E, we have

$$\mu(F) \geq \mu(E) \ \text{(resp. } \mu(F) > \mu(E)\text{)} \ .$$

(ii) For every proper subsheaf F of E, we have

$$\mu(F) \leq \mu(E) \ \text{(resp. } \mu(F) < \mu(E)\text{)} \ .$$

(iii) For every proper quotient sheaf F of E, we have

$$\mu(F) \geq \mu(E) \ \text{(resp. } \mu(F) > \mu(E)\text{)} \ .$$

Remarks:

- Every line bundle on X is stable.
- If $\mathrm{rk}(E)$ and $\deg(E)$ are coprime, the bundle E is semi-stable if and only if it is stable.
- The bundle E is semi-stable (resp. stable) if and only if its dual is.
- Let L be a line bundle on X. Then E is semi-stable (resp. stable) if and only if $E \otimes L$ is.

Let r and d be two integers such that $r \geq 2$. We denote $S(r,d)$ the set of isomorphism classes of semi-stable bundles on X, of rank r and degree d. We denote $S'(r,d)$ the subset of $S(r,d)$ consisting on isomorphism classes of stable bundles.

According to the above, for every integer k, the choice of a line bundle of degree k allows us to define a bijection

$$S(r,d) \longrightarrow S(r,d+kr)$$

inducing a bijection $S'(r,d) \to S'(r,d+kr)$.

On the other hand, if r and d are coprime, we have

$$S(r,d) = S'(r,d).$$

A The Harder–Narasimhan filtration

This is a first justification of the definitions above. Let E be a vector bundle on X.

Proposition 2. *There exists a unique subbundle E_1 of E such that for every subbundle F of E, we have*

$$\mu(F) \leq \mu(E_1)$$

and $\mathrm{rk}(F) \leq \mathrm{rk}(E_1)$ *if* $\mu(F) = \mu(E_1)$.

This subbundle is semistable and it is called the maximal semi-stable subbundle of E.

Lemma 3. *There exists an integer n_0 such that for every subbundle F of E, we have*

$$\mu(F) \leq n_0.$$

Let $\mathcal{O}(1)$ a very ample bundle on X. Then there exists an integer p such that for every integer n greater than p, we have:

$$\mathrm{Hom}(\mathcal{O}(n), E) = \{0\}\,.$$

On the other hand, since a line bundle of degree greater than g has nonzero global sections, for every line bundle L on X of degree greater that $p\deg(\mathcal{O}(1)) + g$, we have

$$\mathrm{Hom}(\mathcal{O}(p), L) \neq \{0\}\,,$$

and thus $\mathrm{Hom}(L, E) = \{0\}$.

Applying the above to the bundles $\wedge^r E$, with $1 \leq r \leq \mathrm{rk}(E) - 1$ it is easy to achieve the proof of Lemma 3.

The existence of a subbundle E_1 of E satisfying the conditions of Proposition 2 follows. It is immediate that E_1 is semistable. It remains to prove its uniqueness. Suppose that E_1' verifies the same properties that E_1, and that $E_1' \neq E_1$.

Let $\pi : E \longrightarrow E/E_1'$ the projection. We have $\pi(E_1) \neq 0$. Let G the subbundle of E/E_1' generated by $\pi(E_1)$. Then, since E_1 semi-stable, we have $\mu(G) \geq \mu(E_1)$.

We have an exact sequence of vector bundles on X:

$$0 \longrightarrow E_1' \longrightarrow \pi^{-1}(G) \longrightarrow G \longrightarrow 0.$$

After the properties of E_1', we have:

$$\mu(\pi^{-1}(G)) < \mu(E_1'), \quad \text{since} \quad \mathrm{rk}(\pi^{-1}(G)) > \mathrm{rk}(E_1')$$

that is

$$\frac{\deg(G) + \deg(E_1')}{\mathrm{rk}(G) + \mathrm{rk}(E_1')} < \frac{\deg(E_1')}{\mathrm{rk}(E_1')},$$

from where we get

$$\mu(G) < \mu(E_1') = \mu(E_1),$$

which is absurd. This proves the uniqueness of E_1 and completes the proof of Proposition 2.

We immediately deduce the

Theorem 4 (Harder–Narasimhan). *There exists a unique filtration of E by vector subbundles,*

$$\{0\} = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{s-1} \subset E_s = E,$$

such that for $1 \leq i \leq s-1$, E_i/E_{i-1} is the maximal semi-stable subbundle of E/E_{i-1}. We call it the Harder–Narasimhan filtration *of E.*

We can now deduce the classification of indecomposable and not semi-stable vector bundles of rank 2 on X. Let Z_0 be the set of isomorphism classes of such bundles. Using Proposition [REF] from Appendix II [REF] and the above theorem, we can prove the

Corollary 5. *Let $f : Z_0 \longrightarrow J \times J$ the mapping defined by $f(E) = (\det(E), L)$, L being the maximal semi-stable subbundle of E. The image of f consists on pairs $(L_1, L_2$ of line bundles such that*

(i) $2\deg(L_1) > \deg(L_2)$

(ii) $h^1(X, L_2^2 \otimes L_1^{-1}) \neq 0.$

Moreover, the fibre of f over a point (L_1, L_2) in its image can be identified with the projective space

$$\mathbb{P}(H^1(X, L_2^2 \otimes L_1^{-1})).$$

B Morphisms of (semi-)stable bundles. Jordan–Hölder theorem

Proposition 6. *Let E and F be semi-stable bundles on X. Then*

a) *If $\mu(F) < \mu(E)$, we have* $\mathrm{Hom}(E, F) = \{0\}$.

b) *If E and F are stable, and $\mu(F) = \mu(E)$, we have* $\mathrm{Hom}(E, F) = \{0\}$ *or* $E \cong F$.

c) *If E is stable, E is simple, that means that their only endomorphisms are homotheties.*

a) Let $f : E \longrightarrow F$ a nonzero morphism. Then we have

$$\mu(\mathrm{im}(f)) \leq \mu(F) < \mu(E)$$

since F is semi-stable. Thus $\ker(f) \neq \{0\}$ and $\mu(\ker(f)) > \mu(E)$, which contradicts the semi-stability of E.

This proves a).

b) With the same notations that a), we have everywhere the big inequalities, and since f is nonzero and E is stable, we have $\ker(f) = \{0\}$ and $\mathrm{im}(f)$ is isomorphic to E. Since $\mu(E) = \mu(F)$ and F is stable, we have $\mathrm{im}(f) = F$, and f is an isomorphism.

This proves b).

c) Let $f : E \longrightarrow E$ be a nonzero morphism. Like in b, we show that it is an isomorphism. Let x a point of X. If λ is an eigenvalue of f_x, $f - \lambda I_X$ is not an isomorphism, and thus $f - \lambda I_X = 0$, and f is a homothety.

This proves c and concludes the proof of Proposition 6.

Corollary 7. *Let E_1 and E_2 semi-stable vector bundles on X such that $\mu(E_1) = \mu(E_2) = \mu$, and E an extension of E_2 by E_1. Then E is semi-stable.*

We have an exact sequence

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0,$$

thus $\mu(E) = \mu$. We are going to show that E is semi-stable.

Let F be a proper subbundle of E, F' its maximal semi-stable subbundle. Let us suppose that $\mu(F) > \mu$, so $\mu(F') > \mu$, and after part a on the Proposition above, we have $\mathrm{Hom}(F', E_2) = \{0\}$, from where we deduce that F' is a subbundle of E_1. But this is absurd since E_1 is semistable.

This proves Corollary 7.

Let μ be a rational number, and C_μ the categorie whose objects are semi-stable vector bundles E on X with $\mu(E) = \mu$, and the morphisms of vector bundles between these bundles.

After Corollary 7, we can define in an obvious way the direct sum of two objects (or two morphisms) of the category C_μ.

Proposition 8. *Let E and F be semi-stable vector bundles on X so that $\mu(E) = \mu(F)$, and $f : E \to F$ a vector bundle morphism.*

Then f has constant rank, $\ker(f)$ *and* $\operatorname{coker}(f)$ *are semi-stable vector bundles and*

$$\mu(\ker(f)) = \mu(\operatorname{coker}(f)) = \mu(E).$$

The morphism f has constant rank if and only if $\operatorname{coker}(f)$ is torsion free. Let T be the torsion subsheaf of $\operatorname{coker}(f)$ and

$$F' = \ker(F \to \operatorname{coker}(f)/T).$$

Since E is semi-stable, we have $\mu(\operatorname{im}(f)) \geq \mu(E)$, and since F is also semi-stable, $\mu(F') \leq \mu(F)$. Thus $F' = \operatorname{im}(f)$ and $\mu(\operatorname{im}(f)) = \mu(E)$. We immediately deduce

$$\mu(E) = \mu(\ker(f)) = \mu(\operatorname{coker}(f)).$$

This concludes the proof of Proposition 8.

From the above we get

Proposition 9. *The category C_μ is Abelian, Artinian and Noetherian.*

We can thus apply the Jordan–Hölder theorem to C_μ, which gives

Theorem 10. *Let E be a semi-stable vector bundle on X. There exists a filtration of E by vector subbundles*

$$0 = E_{p+1} \subset E_p \subset \cdots \subset E_1 \subset E_0 = E$$

such that for $0 \leq i \leq p$, E_i/E_{i+1} is stable and $\mu(E_i/E_{i+1}) = \mu(E)$.

Moreover, the isomorphism class of the bundle $\sum_{i=0}^{p} E_i/E_{i+1}$ depends only on that of E. We denote this isomorphism class by $\operatorname{Gr}(E)$.

The following proposition is related with Proposition 32 [REF].

Proposition 11. *Let E be an object of C_μ. The bundle E is stable if and only if for every object E' of C_μ such that* $\operatorname{Gr}(E') = \operatorname{Gr}(E)$, *we have $E \cong E'$.*

We suppose that E is not stable, and verify the hypotheses of the Proposition. We can write $\operatorname{Gr}(E) = F_1 \oplus F_2$, F_1 being stable and F_2 a direct sum of stable bundles. We easily deduce from part b of Proposition 6 that every exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_1 \oplus F_2 \longrightarrow F_2 \longrightarrow 0$$

$$(\text{resp.} \quad 0 \longrightarrow F_2 \longrightarrow F_1 \oplus F_2 \longrightarrow F_1 \longrightarrow 0)$$

is split. Thus it suffices to show that $h^1(X, F_2^* \otimes F_1) \neq 0$ or $h^1(X, F_1^* \otimes F_2) \neq 0$. Again from part b of Proposition 6, we have

$$h^0(X, F_1^* \otimes F_2) = h^0(X, F_2^* \otimes F_1).$$

We deduce with the theorem of Riemann–Roch, that

$$h^1(X, F_2^* \otimes F_1) - h^1(X, F_1^* \otimes F_2) = 2(\operatorname{rk}(F_1) - \operatorname{rk}(F_2))\mu.$$

If this term is nonzero, one of the integers $h^1(X, F_2^* \otimes F_1)$ and $h^1(X, F_1^* \otimes F_2)$ is nonzero. If it is zero,

$$\chi(X, F_2^* \otimes F_1) = \operatorname{rk}(F_1)\operatorname{rk}(F_2)(1 - g) < 0,$$

and thus $h^1(X, F_2^* \otimes F_1)$ is nonzero.

This concludes the proof of Proposition 11.

II Fine moduli spaces — Coarse moduli spaces

In this section, we pose the problem of the classification of (semi-)stable bundles.

Let Z be a set of isomorphism classes on X, which we will suppose all of the same rank.

Definition 12. A *family of elements of Z parametrized by a Noetherian k-scheme Y* is a locally free sheaf $\mathscr{F}$ over $Y \times_k X$ such that for every closed point y of Y, the isomorphism class of $\mathscr{F}_y$ is an element of Z.

Two families $\mathscr{F}_1$ and $\mathscr{F}_2$ of elements of Z parametrized by Y are said *isomorphic* if there exists an invertible sheaf L on Y such that

$$\mathscr{F}_2 = \mathscr{F}_1 \otimes p_Y^*(L),$$

p_Y denoting the projection $X \times_k Y \longrightarrow Y$. In this case we write $\mathscr{F}_1 \sim \mathscr{F}_2$.

Definition 13. A *fine moduli space* for Z is given by a Noetherian k-scheme Y_0 and a family $\mathscr{F}_0$ of elements of Z parametrized by Y_0 such that for every family $\mathscr{F}$ of elements of Z parametrized by a finite type k-scheme Y, there exists a unique morphism

$$\rho_{\mathscr{F}} : Y \rightarrow Y_0$$

such that

$$\rho_{\mathscr{F}}^*(\mathscr{F}_0) \sim \mathscr{F}.$$

Remark: Functorial interpretation

Let

$$F : k\text{-Sch} \longrightarrow \text{Set}$$

be the functor associating to Y the set of isomorphism classes of families of elements of Z parametrized by Y. Then a fine moduli space for Z is simply the given by a Noetherian k-scheme representing F. Let $\rho : Z \rightarrow Y_0(k)$ the map associating to every element z of Z, represented by a bundle E over $X = X \times \{\text{pt}\}$ the element $\rho_E(\text{pt})$ of $Y_0(k)$. Then we easily see that *ρ is a bijection*. On the other hand it is also immediate that $(Y_0, \mathscr{F}_0)$ is unique up to isomorphism. We call $\mathscr{F}_0$ a *Poincaré bundle*

Definition 14. A *coarse moduli space* for Z is a morphism of functors k-Sch $\rightarrow$ Set

$$\Psi : F \longrightarrow \text{Mor}(-, Y_0),$$

Y_0 being a Noetherian k-scheme satisfying the following conditions:

(i) $\Psi(*) : F(*) \rightarrow Y_0(k)$ is a bijection (where $* = \text{Spec}(k)$, so $F(*) = Z$).

(ii) For every morphism of functors $\Psi_1 : F \rightarrow \text{Mor}(-, Y_0)$, Y_1 being a Noetherian k-scheme, there exists a unique morphism $f : Y_0 \rightarrow Y_1$ such that the following diagram is commutative:

$$\begin{array}{ccc} & & \mathrm{Mor}(-,Y_0) \\ & \overset{\Psi}{\nearrow} & \\ F & & \Big\downarrow \mathrm{Mor}(-,f) \\ & \underset{\Psi_1}{\searrow} & \\ & & \mathrm{Mor}(-,Y_1). \end{array}$$

A fine moduli space for Z defines in an obvious way a coarse moduli space for Z.

It is immediate that a coarse moduli space for Z is unique (up to isomorphism).

In what follows, we will admit the following result: for every pair of integers (r, d) such that $r \geq 2$, the set $S'(r, d)$ of isomorphism classes of stable vector bundles on X of rank r and degree d is nonempty. This will be proven in Section III [REF] [IT SAYS ON THE THIRD PART?].

Recall that $S(r, d)$ denotes the set of isomorphism classes of semi-stable bundles on X, of rank r and degree d. We then have

Proposition 15. *There does not exist a coarse moduli space for $S(r, d)$, if r and d are not coprime.*

(See Theorem 48 [REF], which completes this result).

Since r and d are not coprime, there exist two pairs (r_1, d_1) and (r_2, d_2) of integers such that $r_1 \geq 1$ and $r_2 \geq 2$, $r_1 + r_2 = r$, $d_1 + d_2 = d$ and

$$\frac{d_1}{r_1} + \frac{d_2}{r_2} = \frac{d}{r}.$$

Let E_1 be a stable bundle of rank r_1 and degree d_1 and E_2 a stable bundle of rank r_2 and degree d_2 on X. After Proposition 11, there exists a semi-stable bundle on X of rank r and degree d such that $\mathrm{Gr}(E) = E_1 \oplus E_2$ and not isomorphic to $E_1 \oplus E_2$. We have

Lemma 16. *There exists a bundle E_0 on $\mathbb{A}^1 \times X$, such that*

$$F_0|_{\{0\}\times X} \cong E_1 \oplus E_2 \;\; \textit{and} \;\; F_0|_{\{t\}\times X} \cong E \;\; \textit{if } t \neq 0 \textit{ is an element of } k.$$

We can suppose that we have an exact sequence of vector bundles on X:

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0.$$

We consider the element u of $H^1(\mathbb{A}^1 \times X, p_X^*(E_2^* \otimes E_1))$, where p_X denotes the projection $\mathbb{A}^1 \times X \to X$, image of the element $s \otimes u_0$ of $H^0(\mathbb{A}^1, \mathcal{O}) \otimes H^1(X, E_2^* \otimes E_1)$, s being the section of $\mathcal{O}$ associated to 1_k and u_0 corresponding to the above exact sequence. It is easy to see that the extension F_0 of $p_X^*(E_2)$ by $p_X^*(E_1)$ defined by u defines the hypotheses of Lemma 16.

Let us prove now Proposition 15. We recover the notations from Definition 14. Suppose that there exists a coarse moduli space Y_0 for $S(r, d)$. The bundle F_0 from Lemma 16 is a family of elements of $S(r, d)$ parametrized by $\mathbb{A}^1$. We then have

$$\alpha_{F_0}(0) = \rho(E_1 \oplus E_2) \;\; \text{and} \;\; \alpha_{F_0}(t) = \rho(E) \;\; \text{if} \;\; t \neq 0,$$

but since α_{F_0} is induced by a morphism $\mathbb{A}^1 \to Y_0$, we have $\alpha_{F_0}(0) = \rho(E)$, by continuity, which is absurd.

This achieves the proof of Proposition 15.

Remark: Change of the base field
Let K be a commutative algebraically closed field extension of k, $X_K = X \times_k \mathrm{Spec}(K)$, E a vector bundle on X and $E_K = p^*(E)$, p denoting the projection $X_K \to X$. Then we can show that E_K *is semi-stable if and only if* E *is* (see Chapter 3 [REF] [?]).

We could define a family of semi-stable bundles parametrized by a Noetherian k-scheme Y the following way: it is a locally free sheaf E over $Y \times_k X$ such that for every point y of Y, the vector bundle $(E_y)_{\overline{k(y)}}$ over $X_{\overline{k(y)}}$ ($\overline{k(y)}$ denoting the algebraic closure of $k(y)$), is semi-stable.

In fact these two definitions of a family of semi-stable bundles are equivalent (see Theorem 19′).

III The moduli spaces of (semi-)stable bundles

In this section we sketch the proofs of the following results:

Theorem 17. *Let (r, d) be a pair of integers such that $r \geq 2$. There exists a coarse moduli space for $S'(r, d)$ where the underlying k-scheme is a smooth quasi-projective variety, denoted by $U_s(r, d)$.*

This variety has a natural compactification denoted by $U(r, d)$. The set of k-valued points of $U(r, d)$ is isomorphic to the quotient of $S(r, d)$ by the following equivalence relation: for every pair (E, F) of semi-stable bundles on X of rank r and degree d, E and F are equivalent if and only if $\mathrm{Gr}(E) = \mathrm{Gr}(F)$. The variety $U(r, d)$ is normal.

In the case where r and d are coprime, we have $U(r, d) = U_s(r, d)$.

Theorem 18. *Let (r, d) a pair of coprime integers, with $r \geq 2$. Then there exists a fine moduli space for $S(r, d)$.*

Obviously the underlying k-scheme is $U(r, d)$. After Theorem 18, there exists a Poincaré bundle over $U(r, d)$. We will see on section VI [REF] that there exists a “natural” Poincaré bundle.

We have already admitted that $U_s(r, d) \neq \emptyset$. We can deduce that

$$\dim(U(r, d)) = r^2(g - 1) + 1.$$

Finally we show that semi-stability (resp. stability) is an “open” property.

Theorem 19. *Let Y be a Noetherian k-scheme and W a vector bundle on $Y \times X$. Then the set of points y of $Y(k)$ such that W_y is semi-stable (resp. stable) is an open subset of Y.*

Remark: In fact we can show the following theorem:

Theorem 19′. *Let Y be a Noetherian k-scheme and W a locally free sheaf over $Y \times_k X$. Then the set of points y of Y such that $(W_y)_{\overline{k(y)}}$ is semi-stable (resp. stable) on $X_{\overline{k(y)}}$ is an open subset of Y.*

(See the remark at the end of Section II of the third chapter [REF]).

Remark: The variety $U(r,d)$ also has a "universal property": for every family E of semi-stable vector bundles of rank r and degree d parametrized by a Noetherian k-scheme T, there exists a unique morphism $f : T \to U(r,d)$, such that for every point t of $T(k)$, the point $f(t)$ of $U(r,d)$ is associated to $\mathrm{Gr}(E_t)$.

The first stage on the constrution of the moduli spaces is the search of a family of elements of $S(r,d)$ "containing" all the elements of $S(r,d)$. We achieve this by using the Grothendieck schemes.

A Grothendieck schemes

We easily see that to study $S(r,d)$, we can take d as big as we want. This justifies

Lemma 20. *Let (r,d) a pair of integers such that $r \geq 2$ and $d > r(2g-1)$. Then if E is a semi-stable vector bundle on X, of rank r and degree d, we have:*

(i) *the bundle E is generated by its global sections*

(ii) $h^1(X,E) = 0$.

We then have, after the Theorem of Riemann–Roch,

$$h^0(X,E) = d + r(1-g).$$

In order to prove (ii), we suppose that $h^1(X,E) \neq 0$. By Serre duality, we have $\mathrm{Hom}(E,K) \neq \{0\}$, K denoting the canonical bundle on X. Since E is semi-stable, this implies that

$$r(2g-2) = r\deg(K) \geq d,$$

but this is false by hypothesis, so we have $h^1(X,E) = 0$.

In order to prove (i), it remains to show that for every point x of X, the canonical morphism

$$r_x : H^0(X,E) \longrightarrow E_x$$

is surjective.

If L_x denotes the line bundle on X associated to the divisor x, we have an exact sequence of sheaf morphisms on X

$$0 \longrightarrow E \otimes L_x^{-1} \longrightarrow E \longrightarrow E_x \longrightarrow 0,$$

E_x denoting this time the sheaf on X centered on x and with germ E_x in that point. The long exact sequence associated to the exact sequence above gives

$$H^0(X,E) \xrightarrow{r_x} E_x \longrightarrow H^1(X, E \otimes L_x^{-1}).$$

It thus suffices to prove that $h^1(X, E \otimes L_x^{-1}) = 0$, which results from the fact that

$$\deg(E \otimes L_x^{-1}) = \deg(E) - r > r(2g-2),$$

and from the proof of (ii).

This concludes the proof of Lemma 20.

Let us keep the notations of Lemma 20 and put $p = d + r(1-g)$. It follows from Lemma 20 that the bundle E is isomorphic to a quotient of $\mathcal{O}\otimes k^p$. Moreover, the Hilbert polynomial of E is

$$P(T) = p + r\deg(\mathcal{O}(1))T.$$

In particular, it does not depend of the class of E in $S(r,d)$.

We define now the Grothendieck schemes. (See [8]).

Let $\mathcal{F}$ be a coherent sheaf on X, P_0 an element of $k[T]$ of degree ≤ 1.

A flat family of quotients of $\mathcal{F}$ with Hilbert polynomial P_0 parametrized by a Noetherian k-scheme Y is given by a coherent sheaf $\mathcal{G}$ on $Y\times_k X$, flat over Y, and by a surjective morphism

$$p_X^*(\mathcal{F}) \longrightarrow \mathcal{G}$$

(p_X denoting the projection $Y\times_k X \to X$) such that for every point y of Y the Hilbert polynomial of $\mathcal{G}_y$ over X_y is P_0.

Two such families $\mathcal{G}$ and $\mathcal{G}'$ are said isomorphic if there exists an isomorphism

$$g : \mathcal{G} \longrightarrow \mathcal{G}',$$

such that the diagram

$$\begin{array}{ccc} & & \mathcal{G} \\ & \nearrow & \\ p_X^*(\mathcal{F}) & & \downarrow g \\ & \searrow & \\ & & \mathcal{G}' \end{array}$$

is commutative.

On the other hand, if $f : Y \to Y'$ is a morphism of Noetherian k-schemes and $\mathcal{G}'$ is such a family on Y', we define in an obvious way the family $f^*(\mathcal{G}')$.

We can show that the functor

$$\mathrm{Sch} \longrightarrow \mathrm{Set}$$

associating to Y the set of isomorphism classes of flat families of quotients of $\mathcal{F}$ parametrized by Y is representable by a projective algebraic k-scheme. We denote this k-scheme by $\mathrm{Quot}^{P_0}_{\mathcal{F}/X/k}$.

We put

$$Q = \mathrm{Quot}^{P}_{\mathcal{O}\times k^p/X/k}.$$

It follows from the universal property of Q that there exists over $Q\times X$ a flat family $\mathcal{U}$ of quotients of $\mathcal{O}\otimes k^p$, which is "universal".

Let

$$\rho : p_X^*(\mathcal{O}\otimes k^p) \longrightarrow \mathcal{U}$$

be the canonical morphism.

There exists an open subset R of Q characterized by the following property: for every point q of R, the sheaf $\mathscr{U}_q$ is locally free and the canonical map:

$$H^0(X_q, \mathscr{O} \otimes k^p) \longrightarrow H^0(X_q, \mathscr{U}_q)$$

is an isomorphism. The restriction $\mathscr{V}$ of $\mathscr{U}$ to R is a locally free sheaf of rank r. Endowed with $\mathscr{V}$, R has the following local universal property:

Proposition 21. *Given a Noetherian k-scheme Y and a locally free sheaf F of rank r on $Y \times_k X$, such that for every point y of Y, we have*

(i) *F_y has degree d*

(ii) *F_y is generated by global sections*

(iii) *$h^1(X, F_y) = 0$,*

for every point y_0 of Y, there exists a neighbourhood Y_0 of y_0 and a morphism $f : Y_0 \to R$ such that

$$F|_{Y_0 \times X} \cong f^{\#}(\mathscr{V}).$$

We take as Y_0 an affine neighbourhood of y_0. We easily see that the sheaf $p_{Y*}(F)$ is locally free of rank p on Y (p_Y denoting the projection $Y \times_k X \to Y$). On Y_0 it is then isomorphic to $\mathscr{O}_{Y_0} \otimes k^p$. After (ii), the canonical morphism

$$p_Y^*(p_{Y*}(F)) \longrightarrow F$$

is surjective. Over Y_0, $p_Y^*(p_{Y*}(F))$ is isomorphic to $p_X^*(\mathscr{O} \otimes k^p)$, we thus have a surjective morphism

$$p_X^*(\mathscr{O} \otimes k^p) \longrightarrow F.$$

There exists thus a morphism $f : Y_0 \to Q$ such that $F|_{Y_0 \times X} \cong f^{\#}(\mathscr{U})$. But it is immediate that f takes values on R.

This concludes the proof of Proposition 21.

The group $\mathrm{GL}(p) = \mathrm{Aut}(\mathscr{O} \otimes k^p)$ acts on the sheaf $\mathscr{U}$. We will be happy with making this action explicit: let $\rho : p_X^*(\mathscr{O} \otimes k^p) \to \mathscr{U}$ the canonical morphism, and A an element of $\mathrm{GL}(p)$. We can construct a flat family parametrized by Q in the following way: the underlying sheaf is $\mathscr{U}$ but the surjective morphism is $\rho \circ p_X^*(A^{-1})$. This family defines an automorphism σ_A of Q and an isomorphism $\tau_A : \mathscr{U} \to \sigma_A^{\#}(\mathscr{U})$. For every point z of $Q \times_k X$ and every element u of $\mathscr{U}_z$, we have

$$Au = \tau_A(u).$$

The underlying automorphism of Q is σ_A.

We remark that the action of the subgrop k^*I of $\mathrm{GL}(p)$ on Q is trivial, and as a consequence the action of $\mathrm{GL}(p)$ on Q induces one of $\mathrm{PGL}(p)$. However the action of k^*I on $\mathscr{U}$ is that of k^* by homothety, and thus it is not trivial.

We can make precise the action of $\mathrm{PGL}(p)$ on R:

Proposition 22. (i) *The open set R is* $\mathrm{PGL}(p)$*-invariant.*

(ii) *For every pair (q_1, q_2) of closed points of R the vector bundles over X $\mathscr{U}_{q_1}$ and $\mathscr{U}_{q_2}$ are isomorphic if and only if q_1 and q_2 are in the same orbit of the action of* $\mathrm{PGL}(p)$ *on R.*

(iii) *For every point q of R, the stabilizer of q for the action of* $\mathrm{PGL}(p)$ *is isomorphic to the quotient* $\mathrm{Aut}(\mathscr{U}_q)/k^*I$.

(See Seshadri [36] Prop.6 of chap.II, and Newstead [28] Thm.5.3 p.138 where it is also proven that R is open).

Proposition 23. *The scheme R is an irreducible and smooth quasi-projective variety.*

After Grothendieck, Q is a projective scheme over k. Thus it suffices to prove that R is connected and smooth.

For every closed point q_0 of R, we have a morphism

$$\begin{aligned} \mathrm{PGL}(p) &\longrightarrow R(k) \\ A &\longmapsto A\cdot q \end{aligned}$$

where the image consists on the closed points q of R such that the vector bundles $\mathscr{U}_q$ and $\mathscr{U}_{q_0}$ on X are isomorphic.

In order to show that R is connected, it thus suffices to show that for every pair (q_1, q_2) of closed points of R, there exists a closed point q_0 of R, and a connected component R_1 (resp. R_2) of R gathering the orbits of q_1 and q_0 (resp. q_2 and q_0).

We use the following lemma, due to Serre:

Lemma 24. *Let F be a vector bundle on X generated by global sections. Then there exists an exact sequence*

$$0 \longrightarrow \mathscr{O}\otimes k^{\mathrm{rk}(F)-1} \longrightarrow F \longrightarrow \det(F) \longrightarrow 0.$$

For every point x of X the canonical morphism $H^0(X,F) \to F_x$ is surjective and the set of sections of E that vanish on at least one point of X is a subvariety Y of $H^0(X,F)$ of dimension lower or equal than $h^0(X,F) - \mathrm{rk}(F) + 1$. Thus there exists a vector subspace M of $H^0(X,F)$ of rank $\mathrm{rk}(F) - 1$, not intersecting Y. The canonical morphism of vector bundles on X

$$\mathscr{O}\otimes M \longrightarrow F$$

is injective. Its cokernel is of rank 1 and thus isomorpic to $\det(F)$.

This concludes the proof of Lemma 24.

We consider now the Jacobian $J^{(d)}$ and a Poincaré bundle $\mathscr{L}$ on $J^{(d)} \times X$, we denote E the trivial bundle on $J^{(d)} \times X$, with fibre k^{r-1}. The sheaf $R^1 p_{J*}(\mathrm{Hom}(\mathscr{L}, E))$ on $J^{(d)}$ is locally free of rank $(r-1)(g+d-1)$, p_J denoting the projection $J^{(d)} \times X \to J^{(d)}$. We denote by W this bundle and $\pi : W \longrightarrow J^{(d)}$ the canonical projection. On each point L of $J^{(d)}$, the fibre W_L is $H^1(X, \mathrm{Hom}(\mathscr{L}_L, \mathscr{O}\otimes k^p))$.

There exists a bundle F over $W \times X$, and an exact sequence

$$0 \longrightarrow \pi^{\#}(E) \longrightarrow F \longrightarrow \pi(\mathscr{L}) \longrightarrow 0$$

such that for every point w of W, the restriction of the exact sequence above to $w \times X$:

$$0 \longrightarrow \mathscr{O} \otimes k^p \longrightarrow F_w \longrightarrow \mathscr{L}_{\pi(w)} \longrightarrow 0$$

is associated to the element w of $H^1(X, \underline{\mathrm{Hom}}(\mathscr{L}_{\pi(w)}, \mathscr{O} \otimes k^p))$. This follows from the Remark 2 following the Proposition 2 of Appendix II [REF] and from the fact that for every point L of $J^{(d)}$, we have $h^0(X, \mathscr{L}_L^*) = 0$. The points w of W corresponding to the bundles having the properties of Lemma 20 form an open subset W' of W.

After Lemma 24, there exists a point w_1 (resp. w_2) of W', such that F_{w_1} is isomorphic to $\mathscr{U}_{q_1}$ (resp. $F_{w_2} \ldots$).

After Proposition 21, there exists an open subset W_1 (resp. W_2) of W' and a morphism $f_1 : W_1 \to R$ (resp. $f_2 \ldots$) such that $F|_{W_1 \times X} \cong f_1^{\#}(\mathscr{V})$ (resp. $\ldots$).

But being W irreducible, W_1 and W_2 have nonempty intersection and are connected. This suffices to show our assertion and proves connectedness of R.

For smoothness, it is necessary to use the differential properties of Q. If q is a closed point of R, we have an exact sequence of morphisms of vector bundles on X:

$$0 \longrightarrow C_q \longrightarrow \mathscr{O} \otimes k^p \longrightarrow U_q \longrightarrow 0,$$

C_q denoting the kernel bundle of $\rho|_{\{q\} \times X}$.

After Grothendieck, R is smooth at q if and only if we have $h^1(X, \underline{\mathrm{Hom}}(C_q, \mathscr{U}_q)) = 0$, which follows immediately form the above exact sequence and from the fact that, by the definition of R, we have $h^1(X, \mathscr{U}_q) = 0$.

This concludes the proof of Proposition 23.

Remark: The tangent space to R at q is identified with $\mathrm{Hom}(C_q, \mathscr{U}_q)$. This allows to compute the dimension of R by using the exact sequence above. We find

$$\dim(R) = p^2 + r^2(g-1).$$

From the exact sequence above, we deduce the exact sequence

$$0 \longrightarrow \mathrm{End}(\mathscr{U}_q) \longrightarrow \mathrm{Hom}(\mathscr{O} \otimes k^p, \mathscr{U}_q) \xrightarrow{\ a\ } \mathrm{Hom}(C_q, \mathscr{U}_q) = T_{R,q}$$

The space $\mathrm{Hom}(\mathscr{O} \otimes k^p, \mathscr{U}_q)$ is identified with $M(p)$, the space of $p \times p$ matrices, which is also $T_{\mathrm{PGL}(p),kI}$. The map a is just the tangent map

$$T_{\mathrm{PGL}(p),kI} \longrightarrow T_{R,q},$$

coming form the morphism

$$\begin{aligned} \mathrm{PGL}(p) &\longrightarrow R \\ A &\longmapsto A \cdot q. \end{aligned}$$

B Construction of the moduli spaces

IV The complex case

Chapter 2

Deformation of the moduli varieties of stable bundles

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