The Hitchin fibration: Analogues and generalizations

Guillermo Gallego

Universidad Complutense de Madrid — Instituto de Ciencias Matemáticas

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- ▶ The corresponding line bundle on X_a is given by the eigenvectors.

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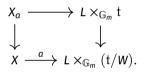
Equivalently, we get an isomorphism

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$$\begin{array}{ccc} X_a & \longrightarrow & L \times_{\mathbb{G}_m} t \\ \downarrow & & \downarrow \\ X & \stackrel{a}{\longrightarrow} & L \times_{\mathbb{G}_m} (t/W). \end{array}$$

Theorem (Donagi-Gaitsgory)

The general fibre $h_{G,L}: \mathcal{M}_{G,L} \to \mathcal{A}_{G,L}$ is a gerbe banded by some sheaf of abelian groups over the associated cameral cover.

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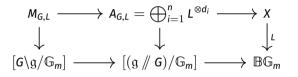
$$[M/G] \longrightarrow M /\!\!/ G.$$

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- We are going to study some examples arising naturally from the theory of Higgs bundles.

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When X is a smooth variety of dimension d and $V = \Omega_X^1$, this is the usual higher-dimensional Hitchin fibration as defined by Simpson.

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These spectral covers are not flat in general! But for dim X ≤ 2 there are flat modifications [Chen-Ngô]. In [G-García-Prada-Narasimhan] we study them in detail for some cases over dim X = 1.

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Example 3: The multiplicative Hitchin fibration

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 $ightharpoonup \mathcal{M} = \{(E, \varphi) : E \to X \text{ G-bundle, } \varphi \in H^0(X', E \times_{\mathsf{Ad}} G) \text{ for } X' = X \setminus \mathsf{finite subset}\}.$

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Idea: Fit the multiplicative Hitchin fibration in a formulation of the form

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- $\blacktriangleright \ \ C_{\mathsf{M}} := \mathsf{M} \ /\!\!/ \ \mathsf{G} = \mathsf{A}_{\mathsf{M}} \times \mathsf{G} \ /\!\!/ \ \mathsf{G}.$

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- $ightharpoonup C_M := M /\!\!/ G = A_M \times G /\!\!/ G.$
- Construct a Hitchin fibration from the sequence

$$[G\backslash M/Z] \longrightarrow [C_M/Z] \longrightarrow [A_M/Z] \longrightarrow \mathbb{B}Z.$$

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- Possible generalization to spherical varieties.

Questions?