The Hitchin fibration: Analogues and generalizations

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- The corresponding line bundle on X_a is given by the eigenvectors.

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Equivalently, we get an isomorphism

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Theorem (Donagi-Gaitsgory)

The general fibre $h_{G,L} : \mathcal{M}_{G,L} \to \mathcal{A}_{G,L}$ is a gerbe banded by some sheaf of abelian groups over the associated cameral cover.

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- There is a natural map

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- We are going to study some examples arising naturally from the theory of Higgs bundles.

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- When X is a smooth variety of dimension d and $V = \Omega_X^1$, this is the usual higher-dimensional Hitchin fibration as defined by Simpson.

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$$\chi_{n,d} : \mathbb{A}^d_k \times \mathrm{Sym}^n \mathbb{A}^d_k \longrightarrow \mathrm{Sym}^n k^d : (x, [x_1, \dots, x_n]) \longmapsto (x - x_1) \dots (x - x_n).$$

- Spectral covers constructed from

$$\begin{array}{ccc} X_a & \longrightarrow & L \times_{\mathrm{GL}_d} \chi_{n,d}^{-1}(0) \\ \downarrow & & \downarrow \mathrm{pr}_2 \\ X & \xrightarrow{a} & V \times_{\mathrm{GL}_d} \mathrm{Sym}^n \mathbb{A}^d_k. \end{array}$$

- These spectral covers are not flat in general! But for $\dim X \leq 2$ there are flat modifications [Chen-Ngô]. In [G–García-Prada–Narasimhan] we study them in detail for some cases over $\dim X = 1$.

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- Construct a Hitchin fibration from the sequence

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- Construct some monoid M_λ as the pullback of this map.
- Recover the multiplicative Hitchin fibration from the sequence

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- Possible generalization to spherical varieties.

Questions?