RESEARCH STATEMENT — GUILLERMO GALLEGO

My research interests lie in the intersection of complex algebraic and differential geometry, mathematical physics, and representation theory. More precisely, my research is centered in studying moduli problems of bundles with additional structure, and in particular in exploring analogues of Hitchin's moduli space of Higgs bundles and of nonabelian Hodge theory, with special interest in the aspects related to Langlands duality and mirror symmetry.

1. Background

1.1. **Higgs bundles, nonabelian Hodge theory and Hitchin fibration.** In his seminal work of 1987 [31], Hitchin introduced what is now called the *moduli space of Higgs bundles*, or simply *Hitchin moduli space*. For a compact Lie group G, a compact Riemann surface C and a topologically trivial principal G-bundle $E \to X$, this moduli space $\mathcal{M}_G(C)$ parametrizes solutions to the *Hitchin equations*; these are pairs (A, Φ) formed by a connection A on E and by a complex Higgs field $\Phi \in \Omega^{1,0}(C, \mathrm{ad}E \otimes \mathbb{C})$ such that

$$\begin{cases} F_A + [\Phi, \Phi^*] = 0, \\ \bar{\partial}_A \Phi = 0. \end{cases}$$

This space is naturally endowed with a *hyperkähler* structure. *Nonabelian Hodge theory* provides a description of the moduli space as a Kähler manifold, when endowed with each one of the complex structures coming from the hyperkähler structure. More precisely, Hitchin (and, in full generality, Simpson [42]) showed that, under one of these structures, this space is biholomorphic to the *Dolbeaut moduli space* $\mathcal{M}_G(C)_{\mathrm{Dol}}$ of polystable topologically trivial $G_{\mathbb{C}}$*-Higgs bundles*, for $G_{\mathbb{C}}$ the complexification of G. Under a different complex structure, it follows from work of Donaldson [11], and more generally of Corlette [8], that the space is biholomorphic to the *Betti moduli space* $\mathcal{M}_G(C)_{\mathrm{B}}$, or *character variety*, which parametrizes conjugacy classes of reductive representations $\pi_1(C) \to G_{\mathbb{C}}$ of the fundamental group of C.

As a natural global analogue of the *Chevalley restriction map*, one obtains the *Hitchin fibration*

$$h_G : \mathcal{M}_G(C)_{\mathrm{Dol}} \to \mathcal{A}_G,$$

originally introduced by Hitchin in another fundamental work, also from 1987 [30]. Hitchin (for the case of G classical) and Donagi-Gaitsgory [9] (in general) showed that the Hitchin fibration is an *algebraic completely integrable Hamiltonian system*, so in particular generic fibres are abelian varieties, and, in fact, they admit a particularly simple description in terms of ramified covers $\tilde{C} \to C$.

1.2. **Mirror symmetry and Langlands duality.** The "best hope" of the geometric Langlands correspondence roughly predicts an equivalence between the derived category of D-modules on the stack $\mathbf{Bun}_{G_{\mathbb{C}}}(C)$ of holomorphic $G_{\mathbb{C}}$-bundles on C and the and the derived category of coherent sheaves on $\mathbf{Flat}_{G^\vee_{\mathbb{C}}}(C)$ the stack of flat $G^\vee_{\mathbb{C}}$-connections on C, for $G^\vee_{\mathbb{C}}$ the Langlands dual group of $G_{\mathbb{C}}$, in a way compatible with the action of Hecke operators. Kapustin and Witten [34] reinterpret the geometric Langlands correspondence as some hyperkähler-enhanced form of homological mirror symmetry for $\mathcal{M}_G(C)$ and its *mirror partner* $\mathcal{M}_{G^\vee}(C)$. In particular, this predicts a correspondence between a certain class of objects, called BAA-branes, supported on holomorphic Lagrangian submanifolds of $\mathcal{M}_G(C)_{\mathrm{Dol}}$ and another class of objects, called BBB-branes, supported on hyperkähler submanifolds of $\mathcal{M}_{G^\vee}(C)$.

Donagi and Pantev [10] studied a "classical limit" of the geometric Langlands correspondence and obtained what physically is understood as SYZ mirror symmetry for the Hitchin fibration. More precisely, they noticed that the Hitchin bases for Langlands dual groups can be identified $\mathcal{A} := \mathcal{A}_G \cong$

$\mathcal{A}_{G^\vee}$ and that, over a generic locus $\mathcal{A}^\diamond \subset \mathcal{A}$, the Hitchin fibres are dual as polarized abelian varieties. This result, combined with the Fourier-Mukai transform, provides an equivalence of categories

$$\mathrm{FM} : \mathrm{D}(\mathrm{Coh}(\mathcal{M}_G(C)_{\mathrm{Dol}}/\mathcal{A}^\diamond)) \longrightarrow \mathrm{D}(\mathrm{Coh}(\mathcal{M}_{G^\vee}(C)_{\mathrm{Dol}}/\mathcal{A}^\diamond)). \tag{1}$$

In more recent times, this correspondence has been used to check the conjectural duality between some pairs of BAA- and BBB-branes over the generic locus.

2. Generalized Hitchin fibrations

2.1. **What?** In his monumental work [39] Ngô assumed a "stacky" point of view on the Hitchin fibration based on the fact that a Higgs bundle can be interpreted as a map from C to the quotient stack $[\mathfrak{g}_{\mathbb{C}}/G_{\mathbb{C}} \times \mathbb{C}^*]$ and lying over the map $C \to B\mathbb{C}^*$ defined by the line bundle K_C. Here, the action of $G_{\mathbb{C}}$ on its Lie algebra is the adjoint action, while the action of $\mathbb{C}^*$ is the homothecy action. The Hitchin fibration is then naturally obtained from the Chevalley restriction map $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{c}_G$, for $\mathfrak{c}_G = \mathrm{Spec}(\mathbb{C}[\mathfrak{g}_{\mathbb{C}}]^{G_{\mathbb{C}}})$.

This point of view then suggests a natural generalization, which considers an affine variety M over an algebrically closed field k, acted on by a reductive algebraic group G over k, and the natural map from the quotient stack $[M/G]$ to the invariant quotient $\mathrm{Spec}(k[M]^G)$ associated to this action. There are several situations in the literature that fit into this description, and a general framework for their study is proposed in unpublished work by Morrissey and Ngô [37] (the interested reader can consult [40]).

2.2. **Why?** There are several reasons for studying generalized Hitchin fibrations in general, the main one being the variety of examples that appear in relation to the study of the "classical" Hitchin fibration. I will state just a few examples:

2.2.1. *Commuting varieties.* Higgs bundles were generalized by Simpson [42] to the context of higher dimensional projective manifolds. A version of the Hitchin fibration in higher dimensions does also exist, and was introduced by Simpson in [43]. One can also construct a similar fibration over a manifold of any dimension (for example in dimension 1), by considering Higgs bundles twisted by a vector bundle, as explained in [15]. From the point of view of generalized Hitchin fibrations, this corresponds to take M equal to the commuting variety of several elements in a Lie algebra, acted on by the adjoint action of the group, and by a general linear group. The main dificulty stems from the fact that spectral covers are generally not flat for this situation. Chen and Ngô [7] studied in detail the case of complex dimension 2, where one can construct a Cohen-Macaulay modification of the spectral cover. In my work with Oscar García-Prada and M.S. Narasimhan [19], we studied the case of complex dimension 1 in [19], giving a more detailed description of the Cohen-Macaulay modification in that case.

2.2.2. *Symmetric pairs.* Versions of Higgs bundles associated to symmetric pairs have been part of the literature ever since the beginning of the discipline. For example, in [32], Hitchin used these kind of pairs to construct higher rank analogues of Teichmüller space inside the character varieties of split real groups. In more recent years, they have received a systematic treatment and, in particular, a version of the nonabelian Hodge correspondence has been developed for them, relating the corresponding moduli space to the character variety for the real form associated to the symmetric pair via the Cartan correspondence. The reader can consult García-Prada's survey [21] for more information about this theory. The Hitchin fibration for symmetric pairs was originally studied in the thesis of Peón-Nieto, and in her subsequent works with García-Prada and Ramanan [24, 25]. A complete description has been given recently by Hameister and Morrissey [28].

2.2.3. *Vinberg pairs.* A version of the Hitchin fibration for Vinberg pairs has been introduced in recent work of García-Prada [22], and a particular case has been also studied in his work with González [23]. The study of the symmetries of this fibration is part of my ongoing research project together with García-Prada and González.

2.2.4. *Hamiltonian G-spaces.* A symplectic manifold M endowed with a Hamiltonian G-action has associated to it a moment map $\mu : M \longrightarrow \mathfrak{g}$ (after choosing an identification of $\mathfrak{g}$ and $\mathfrak{g}^*$). Such a map induces a morphism of stacks $[M/G] \longrightarrow [\mathfrak{g}/G]$, which is Lagrangian in the derived sense of [41], and in turn induces a Lagrangian map to the moduli space of G-Higgs bundles. The case where $M = T^*X$ is the cotangent space of a spherical homogeneous space $X = G/H$ is of particular interest, since in that case there is an isomorphism of stacks $[T^*X/G] \cong [\mathfrak{h}^\perp/H]$. This situation includes in particular the case of symmetric pairs (since symmetric varieties are spherical by the Iwasawa decomposition). A "dual group" $G^\vee_X \subset G^\vee$ associated to any spherical G-variety was constructed by Gaitsgory and Nadler [14], together with an adapted generalization of the geometric Satake correspondence. Conjecturally, the correspondence between BAA- and BBB-branes would match the moduli space of pairs asociated to the spherical pair (G, H) to the moduli space of $G^\vee_X$-Higgs bundles. This conjecture has been verified in some cases, over the generic locus, in the work of Hameister–Luo–Morrissey [29] by tracking which objects are matched under the Donagi–Pantev correspondence (1) and using some of the tools provided by the work of Ben-Zvi, Sakellaridis and Venkatesh on relative Langlands duality [2].

2.2.5. *Reductive monoids.* In the context of multiplicative Higgs bundles, one studies maps into reductive monoids. We will talk in more detail about the multiplicative Hitchin fibration in next section.

2.3. **How? — The regular quotient.** Dealing with generalized Hitchin fibrations is not easy in general, but Morrissey and Ngô [37] provide some general guidelines to do it in a systematic way. The first important task is to determine the *regular quotient.* The regular quotient is a natural rigidification of the quotient stack; it appeared originally in the work of Abramovich, Olsson and Vistoli [1] and was used by Peón-Nieto and García-Prada to the case of symmetric pairs. The general theory of regular quotients is developed in Morrissey–Ngô [37] and has been used in its full power in [28]. Roughly, the regular quotient is a possibly non-separated Deligne–Mumford stack that parametrizes regular orbits. The upshot of using the regular quotient is that, in general, the natural map from a quotient stack of an affine variety by a reductive group into a GIT quotient does not define a gerbe over the GIT quotient, but it does factor through a gerbe over the regular quotient.

2.4. **What about the cameral covers?** The next step towards a complete description of the symmetries of a generalized Hitchin fibration would be to describe the structure of the regular centralizer over the regular quotient. In the context of the classical Hitchin fibration, for classical groups one can make use of the spectral correspondence, while for general groups the Grothendieck-Springer resolution, together with the Chevalley restriction theorem, provides a "Galois" description of the regular quotient in terms of a cameral cover. For the case of the multiplicative Hitchin fibration, as well as for the cases of symmetric pairs and of spherical varieties there exist analogues of the Chevalley restriction theorem and of the Grothendieck-Springer resolution that enable the formulation of a cameral description of the regular centralizer. In general, those tools are not available. Ngô [40] suggested the use of the Luna–Richardson theorem [35] as a possible generalization of the Chevalley restriction theorem. In any case, to the best of my knowledge this direction remains completely unexplored.

2.5. **Langlands duality for generalized Hitchin fibrations.** A generalization of the results of Donagi and Pantev [10] for generalized Hitchin fibrations is desireable, and would provide the right tool for systematically test conjectural matches of BAA- and BBB-branes. In the case in which the regular quotient coincides with the GIT quotient and the regular centralizer admits a Galois description, I claim that such a generalization does exist, and follows from a slight generalization of the results of Donagi and Pantev. This is the content of my forthcoming paper [17]. A generalization for the case where there is a Galois description, but the GIT and regular quotient do not coincide should also be achievable.

3. Multiplicative Higgs bundles

3.1. **Multiplicative Hitchin fibrations.** Given a complex reductive group G, a *multiplicative G-Higgs bundle* on a Riemann surface C is a pair (E, φ) formed by a principal G-bundle $E \to C$ and by a meromorphic section φ of the associated bundle of groups $\mathrm{Ad}(E) = E \times^G G$, where G acts on itself through the adjoint action. More generally, we can consider pairs (G, G') such that G and G' are groups with the same adjoint group and analogously define (G, G')-bundles by replacing $\mathrm{Ad}(E)$ by $E \times^G G'$. If θ is an automorphism of G, we can also consider θ-twisted multiplicative G-Higgs bundles by replacing $\mathrm{Ad}(E)$ by $E \times^{G,\theta} G$. At each of its singularities, the Higgs field φ determines a $G[[t]]$-orbit in the affine Grassmannian $\mathrm{Gr}_G = G((t))/G[[t]]$ and thus a dominant cocharacter of G. A global analogue of the multiplicative version of the Chevalley restriction map $G \to \mathrm{Spec}(\mathbb{C}[G]^G)$ induces the *multiplicative Hitchin fibration.*

The multiplicative Hitchin fibration can be equivalently defined by using the theory of reductive monoids, in a way that is compatible with the framework of generalized Hitchin fibrations. An algebraic monoid is a monoid object in the category of algebraic varieties, and we say that it is reductive if its unit group is a reductive group. The theory of reductive monoids is a particular case of the theory of embeddings of homogeneous spaces (see [44] for a general reference about this theory) and closely parallels the description of toric varieties in terms of combinatorial data. In particular, a reductive monoid M is endowed with a map $M \to A_M$, where A_M is some toric variety determined by M, called the *abelianization* of M, and which encodes the asymptotic behaviour of the group inside the monoid. The prescription of the meromorphic data of a multiplicative G-Higgs bundle determines a reductive monoid M such that the semisimple part of its unit group $M^\times$ is isomorphic to the semisimple part of G, and it also determines a map from C to the quotient stack $[A_M/A_M^\times]$. The abelianization map factors through the natural map from M to its GIT-quotient under the adjoint action of G. Finally, a multiplicative Higgs bundle can be reinterpreted as a map from C to $[M/(G \times A_M^\times)]$ lying over the map $C \to [A_M/A_M^\times]$ determined by the meromorphic data, and the map from M to its GIT-quotient determines the multiplicative Hitchin fibration.

The multiplicative Hitchin fibration was originally introduced in the mathematics literature by Hurtubise and Markman in 2002 [33]. In 2011 Frenkel and Ngô [13] studied some of its properties in the context of a program of geometrization of trace formulas, and they originally proposed the use of the point of view of reductive monoids. Cumulative work of the students of Ngô: Bouthier, Chi and Wang [3, 4, 6, 45, 46], culminated in the proof by Wang of the Fundamental Lemma for the spherical Hecke algebras by exploiting the geometry of the multiplicative Hitchin fibration.

3.2. **Stability conditions, gauge equations and multiplicative nonabelian Hodge theory.** Charbonneau and Hurtubise [5] have defined stability conditions for multiplicative $G_{\mathbb{C}}$-Higgs bundles and proven a Hitchin–Kobayashi type correspondence between polystable multiplicative $G_{\mathbb{C}}$-Higgs bundles over C and singular G-monopoles over the 3-manifold $S^1 \times C$. In particular, the stability condition includes some data corresponding to the positions of the singularities along the circle. An intrinsic interpretation of these parameters in terms uniquely of the Riemann surface C remains open. Therefore, it still makes sense to pose the question of whether a stability condition, a gauge equation, and a Hitchin–Kobayashi type correspondence could be given in a completely intrinsic manner; that is, only considering data depending on C, and not involving the circle nor the extra parameters, at least not a priori.

We expect that this problem fits inside the more general theory of pairs considered by Mundet i Riera [38]. In particular, a problem that stems directly from this one is that of studying the Kähler structures on reductive monoids or on wonderful compactifications of groups, and the associated moment maps for the action of the maximal compact G on them.

Another possible approach consists in considering the natural correspondence existing between multiplicative Higgs bundles and Martens–Thaddeus bundle chains [36], and study the stability conditions associated to that moduli problem.

Imitating the Garland–Murray correspondence between calorons and Kac-Moody monopoles [26], one can stablish a similar correspondence between the Bogomolny equations for a group and the

Hitchin equations for its loop group. This, together with the parallel between the Langlands duality for multiplicative Hitchin system and the Langlands duality for Kac-Moody algebras, suggest that there might exist indeed a deeper connection between multiplicative Higgs bundles and Kac-Moody groups. This exploration is part of an ongoing project with Oscar García-Prada and Jacques Hurtubise.

3.3. **Multiplicative geometric Langlands and Kac-Moody Langlands duality.** In their work of 2018, Elliott and Pestun [12] formulate a series of physical statements that they gather under the name "multiplicative geometric Langlands pseudo-conjectures". Roughly, they predict some version of hyperkähler-enhanced mirror symmetry, in a similar form to the statements of Kapustin–Witten, but for the moduli space of singular monopoles. In the classical limit, however, they conjecture an analogue of the results of Donagi–Pantev for the multiplicative Hitchin fibration, in the case the structure group is simply-laced, and some twisted version of it in the case it is not. In forthcoming work with Benedict Morrissey [20], we give a precise formulation for those statements, and provide a proof.

A particular feature of the Langlands duality for multiplicative Hitchin systems is that it parallels the duality of affine Dynkin diagrams, in the sense that a multiplicative Hitchin fibration associated to a non-simply-laced Dynkin diagram D is dual to a twisted multiplicative Hitchin fibration associated to a pair (D', σ), with D' a Dynkin diagram and σ a diagram automorphism of D' such that the associated folded Dynkin diagram is the dual of D.

3.4. **Relative multiplicative Hitchin fibrations.** The existence of Langlands duality for multiplicative Hitchin fibrations suggests the study of relative versions of the multiplicative Hitchin fibrations, as they might provide examples of branes that could be matched to something else under duality. A simple example of such a relative multiplicative Hitchin fibration is the multiplicative Hitchin fibration for symmetric pairs that I studied in my thesis [16] and in a subsequent paper with Oscar García-Prada [18]. In particular, we showed that the corresponding moduli space appears naturally inside the fixed point locus of the moduli space of multiplicative Higgs bundle, and that it defines a complex Lagrangian submanifold inside of it. The theory of embeddings of symmetric varieties replaces in this context the theory of reductive monoids, and in particular the embedding constructed by Guay [27] plays the same role as the Vinberg monoid for the multiplicative Hitchin fibration. In the final remarks of my thesis [16], I suggest that one should be able to generalize our results to the case of spherical varieties, and that the role of the Vinberg monoid would in general be fulfilled by the spectrum of the Cox ring of the wonderful compactification.

References

[1] Dan Abramovich, Martin Olsson, and Angelo Vistoli, *Tame stacks in positive characteristic*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 4, 1057–1091, DOI 10.5802/aif.2378 (English, with English and French summaries). MR2427954

[2] David Ben-Zvi, Yiannis Sakellaridis, and Akshay Venkatesh, *Relative Langlands duality*, 2024. Preprint, available at https://arxiv.org/abs/2409.04677.

[3] Alexis Bouthier, *Dimension des fibres de Springer affines pour les groupes*, Transform. Groups **20** (2015), no. 3, 615–663, DOI 10.1007/s00031-015-9326-9 (French, with English and French summaries). MR3376144

[4] ———, *La fibration de Hitchin-Frenkel-Ngô et son complexe d'intersection*, Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no. 1, 85–129, DOI 10.24033/asens.2316 (French, with English and French summaries). MR3621427

[5] Benoit Charbonneau and Jacques Hurtubise, *Singular Hermitian-Einstein monopoles on the product of a circle and a Riemann surface*, Int. Math. Res. Not. IMRN **1** (2011), 175–216, DOI 10.1093/imrn/rnq059. MR2755487

[6] Jingren Chi, *Geometry of Kottwitz-Viehmann varieties*, J. Inst. Math. Jussieu **21** (2022), no. 1, 1–65, DOI 10.1017/S1474748019000604. MR4366333

[7] T. H. Chen and B. C. Ngô, *On the Hitchin morphism for higher-dimensional varieties*, Duke Math. J. **169** (2020), no. 10, 1971–2004, DOI 10.1215/00127094-2019-0085. MR4118645

[8] Kevin Corlette, *Flat G-bundles with canonical metrics*, J. Differential Geom. **28** (1988), no. 3, 361–382. MR0965220

[9] R. Y. Donagi and D. Gaitsgory, *The gerbe of Higgs bundles*, Transform. Groups **7** (2002), no. 2, 109–153, DOI 10.1007/s00031-002-0008-z. MR1903115

[10] R. Donagi and T. Pantev, *Langlands duality for Hitchin systems*, Invent. Math. **189** (2012), no. 3, 653–735, DOI 10.1007/s00222-012-0373-8. MR2957305

[11] S. K. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 127–131, DOI 10.1112/plms/s3-55.1.127. MR0887285
[12] Chris Elliott and Vasily Pestun, *Multiplicative Hitchin systems and supersymmetric gauge theory*, Selecta Math. **25** (2019), 1–82.
[13] Edward Frenkel and Bao Châu Ngô, *Geometrization of trace formulas*, Bull. Math. Sci. **1** (2011), no. 1, 129–199, DOI 10.1007/s13373-011-0009-0. MR2823791
[14] Dennis Gaitsgory and David Nadler, *Spherical varieties and Langlands duality*, Mosc. Math. J. **10** (2010), no. 1, 65–137, 271, DOI 10.17323/1609-4514-2010-10-1-65-137 (English, with English and Russian summaries). MR2668830
[15] Guillermo Gallego, *Universal spectral covers and the Hitchin map*, TEMat monográficos, 2: Proceedings of the 3rd BYMAT Conference, 2021, pp. 107–110. Extended abstract. URL: https://temat.es/monograficos/article/view/vol2-p107.
[16] ———, *Multiplicative Higgs bundles, monopoles and involutions*, Ph.D. Thesis, Universidad Complutense de Madrid, 2023. Available at https://guillegallego.xyz/files/GuillermoGallego_Tesis.pdf.
[17] ———, *Duality for tori along ramified coverings of curves*. In preparation.
[18] Guillermo Gallego and Oscar García-Prada, *Multiplicative Higgs bundles and involutions*, Adv. Math. **451** (2024), Paper No. 109789, 56, DOI 10.1016/j.aim.2024.109789. MR4761277
[19] Guillermo Gallego, Oscar García-Prada, and M. S. Narasimhan, *Higgs bundles twisted by a vector bundle*, Internat. J. Math. **35** (2024), no. 9, Paper No. 2441007, 28, DOI 10.1142/S0129167X24410076. MR4772106
[20] Guillermo Gallego and Benedict Morrissey, *Multiplicative Hitchin fibration and Langlands duality*. In preparation.
[21] Oscar García-Prada, *Higgs bundles and higher Teichmüller spaces*, Handbook of Teichmüller theory. Vol. VII, 2020, pp. 239–285.
[22] ———, *Vinberg pairs and Higgs bundles*, Moduli Spaces and Vector Bundles - New Trends, 2024, pp. 199–222.
[23] Oscar García-Prada and Miguel González, *Cyclic Higgs bundles and the Toledo invariant*, 2024. Preprint, available at https://arxiv.org/abs/2403.00415.
[24] Oscar García-Prada and Ana Peón-Nieto, *Abelianization of Higgs bundles for quasi-split real groups*, Transform. Groups **28** (2023), no. 1, 285–325, DOI 10.1007/s00031-021-09658-9. MR4552177
[25] Oscar García-Prada, Ana Peón-Nieto, and S. Ramanan, *Higgs bundles for real groups and the Hitchin-Kostant-Rallis section*, Trans. Amer. Math. Soc. **370** (2018), no. 4, 2907–2953, DOI 10.1090/tran/7363. MR3748589
[26] H. Garland and M. K. Murray, *Kac-Moody monopoles and periodic instantons*, Comm. Math. Phys. **120** (1988), no. 2, 335–351. MR0973538
[27] Nicolas Guay, *Embeddings of symmetric varieties*, Transform. Groups **6** (2001), no. 4, 333–352, DOI 10.1007/BF01237251. MR1870051
[28] Thomas Hameister and Benedict Morrissey, *The Hitchin Fibration for Symmetric Pairs*, 2024. Preprint, available at https://arxiv.org/abs/2407.00961.
[29] Thomas Hameister, Zhilin Luo, and Benedict Morrissey, *Relative Dolbeault Geometric Langlands via the Regular Quotient*, 2024. Preprint, available at https://arxiv.org/abs/2409.15691.
[30] Nigel Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), no. 1, 91–114, DOI 10.1215/S0012-7094-87-05408-1. MR0885778
[31] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126, DOI 10.1112/plms/s3-55.1.59. MR0887284
[32] ———, *Lie groups and Teichmüller space*, Topology **31** (1992), no. 3, 449–473, DOI 10.1016/0040-9383(92)90044-I. MR1174252
[33] J. C. Hurtubise and E. Markman, *Elliptic Sklyanin integrable systems for arbitrary reductive groups*, Adv. Theor. Math. Phys. **6** (2002), no. 5, 873–978 (2003), DOI 10.4310/ATMP.2002.v6.n5.a4. MR1974589
[34] Anton Kapustin and Edward Witten, *Electric-magnetic duality and the geometric Langlands program*, Commun. Number Theory Phys. **1** (2007), no. 1, 1–236, DOI 10.4310/CNTP.2007.v1.n1.a1. MR2306566
[35] D. Luna and R. W. Richardson, *A generalization of the Chevalley restriction theorem*, Duke Math. J. **46** (1979), no. 3, 487–496. MR0544240
[36] Johan Martens and Michael Thaddeus, *Compactifications of reductive groups as moduli stacks of bundles*, Compos. Math. **152** (2016), no. 1, 62–98, DOI 10.1112/S0010437X15007484. MR3453388
[37] Benedict Morrissey and Bao Châu Ngô, *Regular quotients and Hitchin-type fibrations*. Unpublished.
[38] Ignasi Mundet i Riera, *A Hitchin-Kobayashi correspondence for Kähler fibrations*, J. Reine Angew. Math. **528** (2000), 41–80, DOI 10.1515/crll.2000.092. MR1801657
[39] Bao Châu Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. **111** (2010), 1–169, DOI 10.1007/s10240-010-0026-7 (French). MR2653248
[40] ———, *Generalized Hitchin fibration and invariant theory*, ICMAT, April 22, 2023. Video recording available at https://www.youtube.com/watch?v=oV01OKQQXCc.
[41] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 271–328, DOI 10.1007/s10240-013-0054-1. MR3090262
[42] Carlos T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), no. 4, 867–918, DOI 10.2307/1990994. MR0944577

[43] ———, *Moduli of representations of the fundamental group of a smooth projective variety. II*, Inst. Hautes Études Sci. Publ. Math. **80** (1994), 5–79 (1995). MR1320603

[44] Dmitry A. Timashev, *Homogeneous spaces and equivariant embeddings*, Encyclopaedia of Mathematical Sciences, vol. 138, Springer, Heidelberg, 2011. Invariant Theory and Algebraic Transformation Groups, 8. MR2797018

[45] Xiao Wang, *Multiplicative Hitchin Fibration and Fundamental Lemma*, ProQuest LLC, Ann Arbor, MI, 2023. Thesis (Ph.D.)–The University of Chicago. MR4625095

[46] X. Griffin Wang, *Multiplicative Hitchin Fibration and Fundamental Lemma*, 2024. Preprint, available at https://palantiri.griffin.wang/MHFL.pdf.

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