Multiplicative Hitchin fibration and Langlands duality

Guillermo Gallego

Universidad Complutense de Madrid — Instituto de Ciencias Matemáticas

Montreal – 28th February 2024

Joint work (in progress) with Benedict Morrissey

This work is supported by the UCM and Banco Santander under the contract CT63/19-CT64/19

Contents

Contents

Cameral covers

- G a complex semisimple group. $\mathfrak{g}$ its Lie algebra.
- Fix the root datum $\Psi = \Psi_G = (\Phi, \check{\Lambda}, \check{\Phi}, \Lambda)$. $\rightsquigarrow$ T maximal torus, $\mathfrak{t}$ its Lie algebra, W Weyl group.
- A ($\mathfrak{t}$-)cameral cover is a W-Galois cover $\pi : \tilde{X} \to X$ which locally is a pullback of $\mathfrak{t} \to \mathfrak{t}/W$. (cameral curve if $\dim X = 1$).

Cameral covers

- G a complex semisimple group. $\mathfrak{g}$ its Lie algebra.
- Fix the root datum $\Psi = \Psi_G = (\Phi, \check{\Lambda}, \check{\Phi}, \Lambda)$. $\rightsquigarrow$ T maximal torus, $\mathfrak{t}$ its Lie algebra, W Weyl group.
- A ($\mathfrak{t}$-)cameral cover is a W-Galois cover $\pi : \tilde{X} \to X$ which locally is a pullback of $\mathfrak{t} \to \mathfrak{t}/W$. (cameral curve if $\dim X = 1$).
- Let $\pi : \tilde{X} \to X$ be a cameral curve. Put

$$J^1 = J^1_{\pi,\Psi} = \pi_*(\Lambda \otimes \mathcal{O}^\times_{\tilde{X}})^W, \qquad \check{J}^1 = \check{J}^1_{\pi,\Psi} = \pi_*(\check{\Lambda} \otimes \mathcal{O}^\times_{\tilde{X}})^W,$$

$$J = J_{\pi,\Psi} = \left\{\lambda \otimes z \in J^1 : z^{\langle \lambda, \alpha \rangle}|_{\tilde{X}^{s_\alpha}} = 1\right\}, \quad \check{J} = \check{J}_{\pi,\Psi} = \left\{\chi \otimes z \in \check{J}^1 : z^{\langle \chi, \check{\alpha} \rangle}|_{\tilde{X}^{s_{\check{\alpha}}}} = 1\right\}.$$

- Put $\mathscr{P} = \mathscr{P}_{\pi,\Psi} = \mathrm{Bun}_{J/X}$ and $\check{\mathscr{P}} = \check{\mathscr{P}}_{\pi,\Psi} = \mathrm{Bun}_{\check{J}/X}$.

Cameral covers

- G a complex semisimple group. $\mathfrak{g}$ its Lie algebra.
- Fix the root datum $\Psi = \Psi_G = (\Phi, \check{\Lambda}, \check{\Phi}, \Lambda)$. $\rightsquigarrow$ T maximal torus, $\mathfrak{t}$ its Lie algebra, W Weyl group.
- A ($\mathfrak{t}$-)cameral cover is a W-Galois cover $\pi : \tilde{X} \to X$ which locally is a pullback of $\mathfrak{t} \to \mathfrak{t}/W$. (cameral curve if $\dim X = 1$).
- Let $\pi : \tilde{X} \to X$ be a cameral curve. Put

$$J^1 = J^1_{\pi,\Psi} = \pi_*(\Lambda \otimes \mathcal{O}^\times_{\tilde{X}})^W, \qquad \check{J}^1 = \check{J}^1_{\pi,\Psi} = \pi_*(\check{\Lambda} \otimes \mathcal{O}^\times_{\tilde{X}})^W,$$

$$J = J_{\pi,\Psi} = \left\{ \lambda \otimes z \in J^1 : z^{\langle \lambda, \alpha \rangle}|_{\tilde{X}^{s_\alpha}} = 1 \right\}, \quad \check{J} = \check{J}_{\pi,\Psi} = \left\{ \chi \otimes z \in \check{J}^1 : z^{\langle \chi, \check{\alpha} \rangle}|_{\tilde{X}^{s_{\check{\alpha}}}} = 1 \right\}.$$

- Put $\mathscr{P} = \mathscr{P}_{\pi,\Psi} = \mathrm{Bun}_{J/X}$ and $\check{\mathscr{P}} = \check{\mathscr{P}}_{\pi,\Psi} = \mathrm{Bun}_{\check{J}/X}$.
- Donagi-Gaitsgory: Description of $\mathscr{P}$ in terms of $\mathrm{Bun}^W_{T/\tilde{X}}$.

Cameral covers

- G a complex semisimple group. $\mathfrak{g}$ its Lie algebra.
- Fix the root datum $\Psi = \Psi_G = (\Phi, \check{\Lambda}, \check{\Phi}, \Lambda)$. $\rightsquigarrow$ T maximal torus, $\mathfrak{t}$ its Lie algebra, W Weyl group.
- A ($\mathfrak{t}$-)cameral cover is a W-Galois cover $\pi : \tilde{X} \to X$ which locally is a pullback of $\mathfrak{t} \to \mathfrak{t}/W$. (cameral curve if $\dim X = 1$).
- Let $\pi : \tilde{X} \to X$ be a cameral curve. Put

$$J^1 = J^1_{\pi,\Psi} = \pi_*(\Lambda \otimes \mathcal{O}^\times_{\tilde{X}})^W, \qquad \check{J}^1 = \check{J}^1_{\pi,\Psi} = \pi_*(\check{\Lambda} \otimes \mathcal{O}^\times_{\tilde{X}})^W,$$
$$J = J_{\pi,\Psi} = \left\{ \lambda \otimes z \in J^1 : z^{\langle \lambda, \alpha \rangle}|_{\tilde{X}^{s_\alpha}} = 1 \right\}, \quad \check{J} = \check{J}_{\pi,\Psi} = \left\{ \chi \otimes z \in \check{J}^1 : z^{\langle \chi, \check{\alpha} \rangle}|_{\tilde{X}^{s_{\check{\alpha}}}} = 1 \right\}.$$

- Put $\mathscr{P} = \mathscr{P}_{\pi,\Psi} = \mathrm{Bun}_{J/X}$ and $\check{\mathscr{P}} = \check{\mathscr{P}}_{\pi,\Psi} = \mathrm{Bun}_{\check{J}/X}$.
- Donagi-Gaitsgory: Description of $\mathscr{P}$ in terms of $\mathrm{Bun}^W_{T/\tilde{X}}$. $\rightsquigarrow$ $\mathscr{P}$ is a Beilinson 1-motive.

Picard stacks and Beilinson 1-motives

- Picard groupoids are "categories with invertible tensor product". ($\mathrm{Pic}(X)$, $\mathrm{Bun}_T(X)$).
- Picard stacks are stacks "taking values" on Picard groupoids. ($\mathrm{Pic}_X : S \mapsto \mathrm{Pic}(X \times S)$).
- Beilinson 1-motives are locally Picard stacks of the form

$$A \times \mathbb{B}T \times \Gamma,$$

for A an abelian variety, T a torus and Γ a finitely generated abelian group. (Pic_X, $\mathrm{Bun}_{T/X}$).

Picard stacks and Beilinson 1-motives

- Picard groupoids are "categories with invertible tensor product". ($\mathrm{Pic}(X)$, $\mathrm{Bun}_T(X)$).
- Picard stacks are stacks "taking values" on Picard groupoids. ($\mathrm{Pic}_X : S \mapsto \mathrm{Pic}(X \times S)$).
- Beilinson 1-motives are locally Picard stacks of the form

$$A \times \mathbb{B}T \times \Gamma,$$

  for A an abelian variety, T a torus and Γ a finitely generated abelian group. (Pic_X, $\mathrm{Bun}_{T/X}$).
- The dual of a Picard stack $\mathscr{P}$ is the Picard stack

$$\mathscr{P}^\vee := \mathrm{Map}(\mathscr{P}, \mathbb{B}\mathbb{G}_m).$$

Picard stacks and Beilinson 1-motives

- Picard groupoids are "categories with invertible tensor product". ($\mathrm{Pic}(X)$, $\mathrm{Bun}_T(X)$).
- Picard stacks are stacks "taking values" on Picard groupoids. ($\mathrm{Pic}_X : S \mapsto \mathrm{Pic}(X \times S)$).
- Beilinson 1-motives are locally Picard stacks of the form

$$A \times \mathbb{B}T \times \Gamma,$$

  for A an abelian variety, T a torus and Γ a finitely generated abelian group. (Pic_X, $\mathrm{Bun}_{T/X}$).
- The dual of a Picard stack $\mathscr{P}$ is the Picard stack

$$\mathscr{P}^\vee := \mathrm{Map}(\mathscr{P}, \mathbb{B}\mathbb{G}_m).$$

- Beilinson 1-motives are fully-dualizable. As a consequence of this, there is the Fourier-Mukai functor (for $\mathscr{L}_{\mathscr{P}} \to \mathscr{P} \times \mathscr{P}^\vee$ the Poincaré line bundle)

$$\begin{aligned} D^b(\mathrm{QCoh}(\mathscr{P})) &\longrightarrow D^b(\mathrm{QCoh}(\mathscr{P}^\vee)) \\ F &\longmapsto (Rp_2)_*(Lp_1^* F \otimes \mathscr{L}_{\mathscr{P}}). \end{aligned}$$

- For Beilinson 1-motives, this functor is an equivalence of derived categories.

Duality in cameral covers

- Consider the Abel-Jacobi map

$$\begin{aligned} \mathrm{AJ} : \tilde{X} \times \Lambda &\longrightarrow \mathrm{Bun}_{T/\tilde{X}} \\ (x, \lambda) &\longmapsto \mathscr{O}(x\lambda). \end{aligned}$$

  We can compose it with the norm map $\mathrm{Nm} : \mathrm{Bun}_{T/\tilde{X}} \to \mathrm{Bun}^W_{T/\tilde{X}}$. This lifts to $\mathscr{P}$.

- Pulling back line bundles defines $(\mathrm{AJ})^\vee : \mathscr{P}^\vee \to (\mathrm{Pic}^m_{\tilde{X}\times\Lambda})^W = \mathrm{Bun}^W_{\check{T}/\tilde{X}}$. This again lifts to $\check{\mathscr{P}}$.

Duality in cameral covers

- Consider the Abel-Jacobi map

$$\begin{aligned} \mathrm{AJ} : \tilde{X} \times \Lambda &\longrightarrow \mathrm{Bun}_{T/\tilde{X}} \\ (x, \lambda) &\longmapsto \mathcal{O}(x\lambda). \end{aligned}$$

  We can compose it with the norm map $\mathrm{Nm} : \mathrm{Bun}_{T/\tilde{X}} \to \mathrm{Bun}^W_{T/\tilde{X}}$. This lifts to $\mathscr{P}$.

- Pulling back line bundles defines $(\mathrm{AJ})^\vee : \mathscr{P}^\vee \to (\mathrm{Pic}^m_{\tilde{X}\times\Lambda})^W = \mathrm{Bun}^W_{\check{T}/\tilde{X}}$. This again lifts to $\check{\mathscr{P}}$.

Theorem (Hausel-Thaddeus, Donagi-Pantev, Chen-Zhu)

The morphism $\mathscr{P}^\vee \to \check{\mathscr{P}}$ constructed above is an isomorphism of Picard stacks.

Corollary

There is a natural equivalence of derived categories $D^b(QCoh(\mathscr{P})) \to D^b(QCoh(\check{\mathscr{P}}))$.

Contents

The Hitchin fibration

- X smooth complex projective curve. $L \in \mathrm{Pic}(X)$ (usually $L = K_X$).
- $\mathscr{M}_G(X) = \{(E, \varphi) : E \in \mathrm{Bun}_G(X), \varphi \in H^0(X, (E \times^G \mathfrak{g}) \otimes L)\} = \mathrm{Map}_X(X, L \times^{\mathbb{G}_m} [\mathfrak{g}/G])$. (Moduli stack of Higgs bundles).

The Hitchin fibration

- X smooth complex projective curve. $L \in \mathrm{Pic}(X)$ (usually $L = K_X$).
- $\mathscr{M}_G(X) = \{(E, \varphi) : E \in \mathrm{Bun}_G(X), \varphi \in H^0(X, (E \times^G \mathfrak{g}) \otimes L)\} = \mathrm{Map}_X(X, L \times^{\mathbb{G}_m} [\mathfrak{g}/G])$. (Moduli stack of Higgs bundles).
- $\mathscr{A}_G(X) = \bigoplus_{i=1}^r H^0(X, L^{d_i}) = \mathrm{Map}_X(X, L \times^{\mathbb{G}_m} (\mathfrak{t}/W))$, here $d_i = \deg p_i$, where

$$\mathbb{C}[p_1, \dots, p_r] = \mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[\mathfrak{g}]^G.$$

(Hitchin base).

The Hitchin fibration

- X smooth complex projective curve. $L \in \mathrm{Pic}(X)$ (usually $L = K_X$).
- $\mathscr{M}_G(X) = \{(E,\varphi) : E \in \mathrm{Bun}_G(X), \varphi \in H^0(X, (E \times^G \mathfrak{g}) \otimes L)\} = \mathrm{Map}_X(X, L \times^{\mathbb{G}_m} [\mathfrak{g}/G])$. (Moduli stack of Higgs bundles).
- $\mathscr{A}_G(X) = \bigoplus_{i=1}^r H^0(X, L^{d_i}) = \mathrm{Map}_X(X, L \times^{\mathbb{G}_m} (\mathfrak{t}/W))$, here $d_i = \deg p_i$, where

$$\mathbb{C}[p_1, \dots, p_r] = \mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[\mathfrak{g}]^G.$$

  (Hitchin base).
- The Hitchin fibration is the map

$$h_G : \mathscr{M}_G(X) \longrightarrow \mathscr{A}_G(X)$$

  naturally induced from the Chevalley restriction map $\mathfrak{g} \to \mathfrak{g} /\!\!/ G$.
- Problem: Study the fibres $\mathscr{M}_{G,a} = h_G^{-1}(a)$.

The symmetries

- Take the centralizer group scheme $I_G \to \mathfrak{g}$, that is

$$I_{G,x} = \left\{ g \in G : \mathrm{Ad}_g x = x \right\}.$$

It is a smooth group scheme over $\mathfrak{g}^{\mathrm{reg}} \subset \mathfrak{g}$.

The symmetries

- Take the centralizer group scheme $I_G \to \mathfrak{g}$, that is

$$I_{G,x} = \left\{ g \in G : \mathrm{Ad}_g x = x \right\}.$$

It is a smooth group scheme over $\mathfrak{g}^{\mathrm{reg}} \subset \mathfrak{g}$.

Theorem (Ngô)

$I_G|_{\mathfrak{g}^{reg}}$ descends to a group scheme $J_G \to \mathfrak{g} /\!\!/ G$, called the regular centralizer.

The symmetries

- Take the centralizer group scheme $I_G \to \mathfrak{g}$, that is

$$I_{G,x} = \left\{ g \in G : \mathrm{Ad}_g x = x \right\}.$$

It is a smooth group scheme over $\mathfrak{g}^{\mathrm{reg}} \subset \mathfrak{g}$.

Theorem (Ngô)

$I_G|_{\mathfrak{g}^{reg}}$ descends to a group scheme $J_G \to \mathfrak{g} /\!\!/ G$, called the regular centralizer.

- Given $a \in \mathscr{A}_G(X)$, we let $J_{G,a} = a^* J_G$ be the pullback along the natural map $a : X \to [(\mathfrak{g} /\!\!/ G)/\mathbb{G}_m]$. Put $\mathscr{P}_{G,a} = \mathrm{Bun}_{J_{G,a}/X}$.

Theorem (Donagi-Gaitsgory, Ngô)

$\mathscr{M}^{reg}_{G,a}$ is a $\mathscr{P}_{G,a}$-torsor.

The symmetries

- Take the centralizer group scheme $I_G \to \mathfrak{g}$, that is

$$I_{G,x} = \left\{g \in G : \mathrm{Ad}_g x = x\right\}.$$

It is a smooth group scheme over $\mathfrak{g}^{\mathrm{reg}} \subset \mathfrak{g}$.

Theorem (Ngô)

$I_G|_{\mathfrak{g}^{reg}}$ descends to a group scheme $J_G \to \mathfrak{g} /\!\!/ G$, called the regular centralizer.

- Given $a \in \mathscr{A}_G(X)$, we let $J_{G,a} = a^* J_G$ be the pullback along the natural map $a : X \to [(\mathfrak{g} /\!\!/ G)/\mathbb{G}_m]$. Put $\mathscr{P}_{G,a} = \mathrm{Bun}_{J_{G,a}/X}$.

Theorem (Donagi-Gaitsgory, Ngô)

$\mathscr{M}^{reg}_{G,a}$ is a $\mathscr{P}_{G,a}$-torsor. $\rightsquigarrow$ Study $\mathscr{P}_{G,a}$.

The cameral description

Theorem (Donagi-Gaitsgory, Ngô)

$\mathscr{P}_{G,a} = \mathscr{P}_{\pi_a,\Psi_G}$, *where* $\pi_a : \tilde{X}_a \to X$ *is the cameral cover obtained as the pullback*

$$\begin{array}{ccc} \tilde{X}_a & \longrightarrow & L \times^{\mathbb{G}_m} \mathfrak{t} \\ \big\downarrow \pi_a & & \big\downarrow \\ X & \xrightarrow{a} & L \times^{\mathbb{G}_m} (\mathfrak{t}/W). \end{array}$$

Langlands duality of Hitchin fibrations

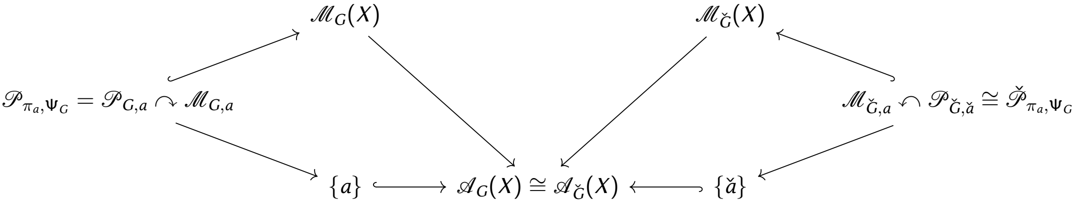
- A W-invariant metric on $\mathfrak{t}$ induces an isomorphism $\mathfrak{t}/W \to \mathfrak{t}^*/W$.
- In turn this gives an isomorphism $\mathscr{A}_G(X) \to \mathscr{A}_{\check{G}}(X)$, $a \mapsto \check{a}$, and we have

$$\mathscr{P}_{\check{G},\check{a}} = \mathscr{P}_{\pi_{\check{a}},\Psi_{\check{G}}} \cong \check{\mathscr{P}}_{\pi_a,\Psi_G}.$$

Langlands duality of Hitchin fibrations

- A W-invariant metric on $\mathfrak{t}$ induces an isomorphism $\mathfrak{t}/W \to \mathfrak{t}^*/W$.
- In turn this gives an isomorphism $\mathscr{A}_G(X) \to \mathscr{A}_{\check{G}}(X)$, $a \mapsto \check{a}$, and we have

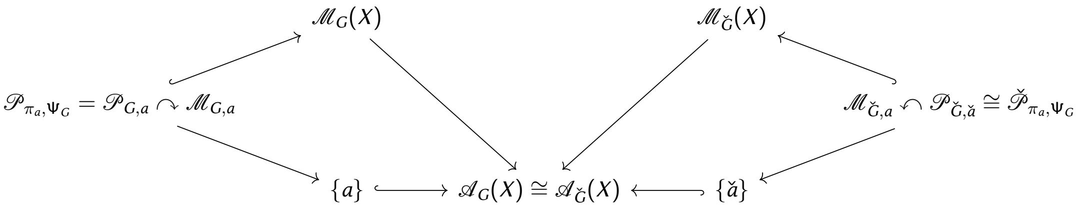
$$\mathscr{P}_{\check{G},\check{a}} = \mathscr{P}_{\pi_{\check{a}},\Psi_{\check{G}}} \cong \check{\mathscr{P}}_{\pi_a,\Psi_G}.$$



Langlands duality of Hitchin fibrations

- A W-invariant metric on $\mathfrak{t}$ induces an isomorphism $\mathfrak{t}/W \to \mathfrak{t}^*/W$.
- In turn this gives an isomorphism $\mathscr{A}_G(X) \to \mathscr{A}_{\check{G}}(X)$, $a \mapsto \check{a}$, and we have

$$\mathscr{P}_{\check{G},\check{a}} = \mathscr{P}_{\pi_{\check{a}},\Psi_{\check{G}}} \cong \check{\mathscr{P}}_{\pi_a,\Psi_G}.$$



Corollary ("Classical limit" of Geometric Langlands)

There is a natural equivalence of derived categories $D^b(QCoh(\mathscr{P}_{G,a})) \to D^b(QCoh(\mathscr{P}_{\check{G},\check{a}}))$.

Contents

Multiplicative Higgs bundles

- X smooth complex projective curve. Assume G is semisimple simply-connected.
- Let G_0 be another complex semisimple group with $G_0^{\mathrm{ad}} = G^{\mathrm{ad}}$.
- $\mathscr{M}_{G_0,G}(X) =$
  $\left\{(E,\varphi) : E \in \mathrm{Bun}_{G_0}(X), \varphi \in \Gamma(X \setminus \{x_1,\dots,x_n\}, E \times^G G) \text{ for some } x_1,\dots,x_n \in X\right\}$.
  (Moduli stack of multiplicative (G_0, G)-Higgs bundles).

Multiplicative Higgs bundles

- X smooth complex projective curve. Assume G is semisimple simply-connected.
- Let G_0 be another complex semisimple group with $G_0^{\mathrm{ad}} = G^{\mathrm{ad}}$.
- $\mathscr{M}_{G_0,G}(X) =$
  $\left\{(E,\varphi) : E \in \mathrm{Bun}_{G_0}(X), \varphi \in \Gamma(X \setminus \{x_1,\dots,x_n\}, E \times^G G) \text{ for some } x_1,\dots,x_n \in X\right\}$.
  (Moduli stack of multiplicative (G_0, G)-Higgs bundles).
- z local coordinate around z_i.
  $\varphi|_{\mathbb{D}_{x_i}} \rightsquigarrow$ well defined element of $G_0[[z]]\backslash G((z))/G_0[[z]] \cong \Lambda^+ \cong \Lambda/W$.
- Idea: Prescribe singularities by choosing $D = \sum_{i=1}^n \lambda_i x_i$ a Λ^+-valued divisor on X.

The multiplicative Hitchin fibration

- Let $\varpi_1, \ldots, \varpi_r$ be the fundamental weights of G.
- Multiplicative version of Chevalley: $G /\!\!/ \mathrm{Ad}(G) \cong T/W = \mathrm{Spec}(\mathbb{C}[p_1, \ldots, p_r])$, for $p_i = \mathrm{tr}(\rho_i)$, and ρ_i the fundamental representation of weight ϖ_i. (G simply-connected!)

Multiplicative Hitchin fibration (Hurtubise-Markman)

$$h_{G_0,G} : \mathscr{M}_{G_0,G,D}(X) \longrightarrow \mathscr{A}_{G,D}(X) = \bigoplus_{i=1}^{r} H^0(X, \mathscr{O}(\langle D, \varpi_i \rangle))$$
$$(E, \varphi) \longmapsto (p_1(\varphi), \ldots, p_r(\varphi)).$$

- Study the fibres $h^{-1}_{G_0,G,D}(a) = \mathscr{M}_{G_0,G,D,a}$.

The monoid POV (Frenkel-Ngô, Bouthier, J. Chi, G. Wang)

- Idea: Construct a partial compactification $G^D \to X$ of G over X, depending on the Λ^+-valued divisor $D = \sum_{i=1}^n \lambda_i x_i$, such that

$$\mathscr{M}_{G_0,G,D}(X) = \mathrm{Map}_X(X, [G^D/G_0]), \qquad \mathscr{A}_{G,D}(X) = \mathrm{Map}_X(X, G^D /\!\!/ \mathrm{Ad}(G)).$$

The monoid POV (Frenkel-Ngô, Bouthier, J. Chi, G. Wang)

- Idea: Construct a partial compactification $G^D \to X$ of G over X, depending on the Λ^+-valued divisor $D = \sum_{i=1}^n \lambda_i x_i$, such that

$$\mathscr{M}_{G_0,G,D}(X) = \mathrm{Map}_X(X, [G^D/G_0]), \qquad \mathscr{A}_{G,D}(X) = \mathrm{Map}_X(X, G^D /\!\!/ \mathrm{Ad}(G)).$$

- Consider the Vinberg monoid $\mathrm{Vin}(G)$. This is a natural compactification of $(G \times T)/Z_G$, endowed with a natural map $\mathrm{Vin}(G) \to \mathbb{A}$, for $\mathbb{A} \cong \mathbb{A}^r$ the T-toric variety defined by

$$\lim_{z\to 0} z^\lambda \in \mathbb{A} \Leftrightarrow \lambda \in \Lambda^+.$$

The T-action is $t \cdot (a_1, \ldots, a_r) = (t^{\alpha_1} a_1, \ldots, t^{\alpha_r} a_r)$.

The monoid POV (Frenkel-Ngô, Bouthier, J. Chi, G. Wang)

- Idea: Construct a partial compactification $G^D \to X$ of G over X, depending on the Λ^+-valued divisor $D = \sum_{i=1}^n \lambda_i x_i$, such that

$$\mathscr{M}_{G_0,G,D}(X) = \mathrm{Map}_X(X, [G^D/G_0]), \qquad \mathscr{A}_{G,D}(X) = \mathrm{Map}_X(X, G^D /\!\!/ \mathrm{Ad}(G)).$$

- Consider the Vinberg monoid $\mathrm{Vin}(G)$. This is a natural compactification of $(G \times T)/Z_G$, endowed with a natural map $\mathrm{Vin}(G) \to \mathbb{A}$, for $\mathbb{A} \cong \mathbb{A}^r$ the T-toric variety defined by

$$\lim_{z \to 0} z^\lambda \in \mathbb{A} \Leftrightarrow \lambda \in \Lambda^+.$$

The T-action is $t \cdot (a_1, \ldots, a_r) = (t^{\alpha_1} a_1, \ldots, t^{\alpha_r} a_r)$.

- G^D is obtained as the pullback

$$\begin{array}{ccc} G^D & \longrightarrow & [\mathrm{Vin}(G)/T] \\ \downarrow & & \downarrow \\ X & \xrightarrow{(\mathscr{O}(D), \sigma_{\mathscr{O}(D)})} & [\mathbb{A}/T]. \end{array}$$

The symmetries

- Take the centralizer group scheme $I_{G_0,G,D} \to G^D$, that is

$$I_{G_0,G,D,h} = \left\{ g \in G_0 : ghg^{-1} = h \right\}.$$

It is a smooth group scheme over $G^D_{\mathrm{reg}} \subset G^D$.

Theorem (Bouthier–J. Chi–G. Wang)

$I_{G_0,G,D}|_{G^D_{reg}}$ descends to a group scheme $J_{G_0,G,D} \to G^D /\!\!/ Ad(G)$, called the regular centralizer. (G simply-connected!)

The symmetries

- Take the centralizer group scheme $I_{G_0,G,D} \to G^D$, that is

$$I_{G_0,G,D,h} = \left\{g \in G_0 : ghg^{-1} = h\right\}.$$

It is a smooth group scheme over $G^D_{\mathrm{reg}} \subset G^D$.

Theorem (Bouthier–J. Chi–G. Wang)

$I_{G_0,G,D}|_{G^D_{reg}}$ descends to a group scheme $J_{G_0,G,D} \to G^D /\!\!/ \mathrm{Ad}(G)$, called the regular centralizer. (G simply-connected!)

- Given $a \in \mathscr{A}_{G,D}(X)$, we let $J_{G_0,G,D,a} = a^* J_{G_0,G,D}$ be the pullback along the map $a : X \to G^D /\!\!/ \mathrm{Ad}(G)$. Put $\mathscr{P}_{G_0,G,D,a} = \mathrm{Bun}_{J_{G_0,G,D,a}/X}$.

Theorem (Bouthier–J. Chi–G. Wang)

$\mathscr{M}_{G_0,G,D,a} = h^{-1}_{G_0,G,D}(a)$ is a $\mathscr{P}_{G_0,G,D,a}$-torsor.

The cameral description

Theorem (J. Chi–G. Wang)

$\mathscr{P}_{G_0,G,D,a} = \mathscr{P}_{\pi_a,\Psi_{G_0}}$, *where* $\pi_a : \tilde{X}_a \to X$ *is the cameral cover obtained as the pullback*

$$\begin{array}{ccc} \tilde{X}_a & \longrightarrow & T^D \\ \downarrow{\scriptstyle \pi_a} & & \downarrow \\ X & \xrightarrow{a} & T^D/W \cong G^D /\!\!/ Ad(G). \end{array}$$

- This is indeed a cameral cover, locally isomorphic to $\mathfrak{t} \to \mathfrak{t}/W$.

Contents

The simply-laced case

- Let G be a semisimple simply-connected group. Let G_0, G_1 be semisimple groups with $G_0^{\mathrm{ad}} = G_1^{\mathrm{ad}} = G^{\mathrm{ad}}$.
- Let D be a Λ^+-valued divisor and consider the stacks $\mathscr{M}_0 = \mathscr{M}_{G_0,G,D}(X)$ and $\mathscr{M}_1 = \mathscr{M}_{G_1,G,D}(X)$.
- Both stacks fiber over $\mathscr{A}_{G,D}(X)$. Take $a \in \mathscr{A}_{G,D}(X)$ and consider the fibres $\mathscr{M}_{0,a}$ and $\mathscr{M}_{1,a}$.
- These fibres are torsors over $\mathscr{P}_{0,a} = \mathscr{P}_{\pi_a,\Psi_{G_0}}$ and $\mathscr{P}_{1,a} = \mathscr{P}_{\pi_a,\Psi_{G_1}}$, respectively.

The simply-laced case

- Let G be a semisimple simply-connected group. Let G_0, G_1 be semisimple groups with $G_0^{\mathrm{ad}} = G_1^{\mathrm{ad}} = G^{\mathrm{ad}}$.
- Let D be a Λ^+-valued divisor and consider the stacks $\mathscr{M}_0 = \mathscr{M}_{G_0,G,D}(X)$ and $\mathscr{M}_1 = \mathscr{M}_{G_1,G,D}(X)$.
- Both stacks fiber over $\mathscr{A}_{G,D}(X)$. Take $a \in \mathscr{A}_{G,D}(X)$ and consider the fibres $\mathscr{M}_{0,a}$ and $\mathscr{M}_{1,a}$.
- These fibres are torsors over $\mathscr{P}_{0,a} = \mathscr{P}_{\pi_a,\Psi_{G_0}}$ and $\mathscr{P}_{1,a} = \mathscr{P}_{\pi_a,\Psi_{G_1}}$, respectively.
- If G is simply-laced, we can take G_0 and G_1 to be Langlands dual, and thus

$$\mathscr{P}_{1,a} = \mathscr{P}_{\pi_a,\Psi_{G_1}} = \mathscr{P}_{\pi_a,\Psi_{G_0}^\vee} = \check{\mathscr{P}}_{\pi_a,\Psi_{G_0}} \cong \mathscr{P}_{0,a}^\vee.$$

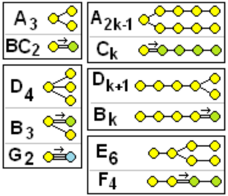
Theorem (G.–Morrissey)

If G is simply-laced, there is a natural equivalence of derived categories

$$D^b(QCoh(\mathscr{P}_{G_0,G,D,a})) \to D^b(QCoh(\mathscr{P}_{\check{G}_0,G,D,a})).$$

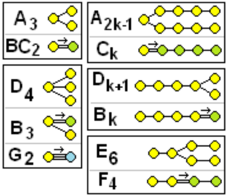
Folding

- Non simply-laced non self-dual groups can be obtained from simply-laced ones by means of folding.
- Let G be a semisimple group, $\Psi_G = (\Phi, \check{\Lambda}, \check{\Phi}, \Lambda)$, and $\theta \in \mathrm{Aut}(G)$ an automorphism stabilizing Ψ_G.
- From Φ and θ one can construct the folded root system Φ_θ.
- The root datum $\Psi_{G_\theta} = (\Phi_\theta, \check{\Lambda}^\theta, \Phi_\theta^\vee, \Lambda_\theta)$ defines a group G_θ called the coinvariant group.



Folding

- Non simply-laced non self-dual groups can be obtained from simply-laced ones by means of folding.
- Let G be a semisimple group, $\Psi_G = (\Phi, \check{\Lambda}, \check{\Phi}, \Lambda)$, and $\theta \in \mathrm{Aut}(G)$ an automorphism stabilizing Ψ_G.
- From Φ and θ one can construct the folded root system Φ_θ.
- The root datum $\Psi_{G_\theta} = (\Phi_\theta, \check{\Lambda}^\theta, \Phi_\theta^\vee, \Lambda_\theta)$ defines a group G_θ called the coinvariant group.



- G_θ is Langlands dual to $(\check{G}^{\check{\theta}})^0$.
- $G_\theta /\!\!/ \mathrm{Ad}(G_\theta)$ is isomorphic to $G/_\theta G$, with G acting on itself through θ-twisted conjugation. [Moehrdieck].

The twisted multiplicative Hitchin fibration

- X smooth complex projective curve. Assume G is semisimple simply-connected.
- $\mathscr{M}_{G,\theta}(X) =$
  $\left\{(E,\varphi) : E \in \mathrm{Bun}_G(X), \varphi \in \Gamma(X \setminus \{x_1,\ldots,x_n\}, E \times^{(G,\theta)} G) \text{ for some } x_1,\ldots,x_n \in X\right\}$.
  (Moduli stack of θ-twisted multiplicative G-Higgs bundles).
- Singularities work the same as in the untwisted case.

The twisted multiplicative Hitchin fibration

- X smooth complex projective curve. Assume G is semisimple simply-connected.
- $\mathscr{M}_{G,\theta}(X) =$
  $\left\{(E,\varphi) : E \in \mathrm{Bun}_G(X), \varphi \in \Gamma(X \setminus \{x_1,\dots,x_n\}, E \times^{(G,\theta)} G) \text{ for some } x_1,\dots,x_n \in X\right\}$.
  (Moduli stack of θ-twisted multiplicative G-Higgs bundles).
- Singularities work the same as in the untwisted case.
- Let $\eta_1,\dots,\eta_s$ be the fundamental weights of the folded root system Φ_θ.
- $G /\!\!/_\theta G \cong T_0^\theta\theta/\tilde{W} = \mathrm{Spec}(\mathbb{C}[q_1,\dots,q_s])$, for $q_i = \mathrm{tr}(\tilde{\rho}_{\eta_i})$, $\tilde{W} = ((1-\theta)(T) \cap T_0^\theta) \rtimes W^\theta$.
- The θ-twisted multiplicative Hitchin fibration

$$h_{G,\theta,D} : \mathscr{M}_{G,\theta,D}(X) \longrightarrow \mathscr{A}_{G,\theta,D}(X) = \bigoplus_{i=1}^{s} H^0(X, \mathscr{O}(\langle D, \eta_i\rangle))$$

$$(E,\varphi) \longmapsto (q_1(\varphi),\dots,q_s(\varphi)).$$

The twisted multiplicative Hitchin fibration

- X smooth complex projective curve. Assume G is semisimple simply-connected.
- $\mathscr{M}_{G,\theta}(X) =$
  $\{(E,\varphi) : E \in \mathrm{Bun}_G(X), \varphi \in \Gamma(X \setminus \{x_1,\dots,x_n\}, E \times^{(G,\theta)} G) \text{ for some } x_1,\dots,x_n \in X\}$.
  (Moduli stack of θ-twisted multiplicative G-Higgs bundles).
- Singularities work the same as in the untwisted case.
- Let $\eta_1,\dots,\eta_s$ be the fundamental weights of the folded root system Φ_θ.
- $G /\!\!/_\theta G \cong T_0^\theta \theta/\tilde{W} = \mathrm{Spec}(\mathbb{C}[q_1,\dots,q_s])$, for $q_i = \mathrm{tr}(\tilde{\rho}_{\eta_i})$, $\tilde{W} = ((1-\theta)(T) \cap T_0^\theta) \rtimes W^\theta$.
- The θ-twisted multiplicative Hitchin fibration

$$h_{G,\theta,D} : \mathscr{M}_{G,\theta,D}(X) \longrightarrow \mathscr{A}_{G,\theta,D}(X) = \bigoplus_{i=1}^{s} H^0(X, \mathscr{O}(\langle D, \eta_i \rangle))$$
$$(E,\varphi) \longmapsto (q_1(\varphi),\dots,q_s(\varphi)).$$

- The θ-twisted multiplicative Hitchin fibration admits a monoid POV and a cameral description in terms of a cameral curve for $\mathfrak{t}_\theta \to \mathfrak{t}_\theta/W$. The centralizers are locally T^θ.

General duality

- Assume that H is a semisimple simply-connected non-simply laced group with $H = G_\theta$, for some semisimple simply-connected simply laced group G. Note that $H^{\mathrm{ad}} = \check{G}_{\check{\theta}}$.
- $\mathscr{M}_{G,\theta,D}(X)$ and $\mathscr{M}_{H^{\mathrm{ad}},H,D}(X)$ both fiber over $\mathscr{A}_{G,\theta,D}(X) = \mathscr{A}_{H,D}(X)$.
- The fibres are torsors over $\check{\mathscr{P}}_{\pi_a,\Psi_{H^{\mathrm{ad}}}}$ and over $\mathscr{P}_{\pi_a,\Psi_{H^{\mathrm{ad}}}}$, respectively.

Theorem (G.–Morrissey)

There is a natural equivalence of derived categories

$$D^b(QCoh(\mathscr{P}_{G,\theta,D,a})) \to D^b(QCoh(\mathscr{P}_{H^{ad},H,D,a})).$$

Future work

- Statements for very flat monoids.
- G non-simply connected ($\rightsquigarrow$ regular quotients)
- Multiplicative (G_0, G)-Higgs bundles, for general G_0, G isogenous. Twisted version?
- Mirror symmetry. Branes.
- Geometric Langlands.
- Connections to 6d SCFT. [Z. Duan–K. Lee–J. Nahmgoong–X. Wang].

Merci beaucoup!