Some facts on local algebra

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1 Cohen-Macaulay modules

Let $(A, \mathfrak{m})$ be a Noetherian local ring and $\kappa = A/\mathfrak{m}$ its residue field.

Definition 1.1. Let M be an A-module. A *regular M-sequence* is a sequence $(a_1, \dots, a_p)$ of elements of $\mathfrak{m}$ such that for every i, $1 \leq i \leq p$, a_i is not a zero-divisor of $M/(a_1, \dots, a_{i-1})M$.

Recall that $a \in A$ is a zero-divisor of an A-module M if there exists some $x \in M$ such that $ax = 0$. In other words, if $a \in A$ is not a zero-divisor of M, then the endomorphism

$$\begin{aligned} a : M &\longrightarrow M \\ x &\longmapsto ax \end{aligned}$$

is injective.

If $(a_1, \dots, a_p)$ is a regular M-sequence and $b \in \mathfrak{m}$ is another element of the maximal ideal, then for any morphism $f : \kappa \to M/(a_1, \dots, a_p)M$, the sequence

$$\kappa \xrightarrow{f} M/(a_1, \dots, a_p)M \xrightarrow{b} M/(a_1, \dots, a_p)M$$

is exact, since $bf(a + \mathfrak{m}) = f(\mathfrak{m}) = 0$. Therefore, if b is not a zero-divisor of $M/(a_1, \dots, a_p)M$, the morphism f is identically 0. We then conclude that if the regular M-sequence $(a_1, \dots, a_p)$ can be extended to a longer sequence $(a_1, \dots, a_p, b)$, then

$$\mathrm{Hom}_A(\kappa, M/(a_1, \dots, a_p)M) = 0.$$

Proposition 1.2. *Let M be an A-module and $(a_1, \dots, a_p)$ a regular M-sequence. Then*

$$\mathrm{Ext}^i_A(\kappa, M) = \begin{cases} 0, & \text{if } i < p, \\ \mathrm{Hom}_A(\kappa, M/(a_1, \dots, a_p)M) \neq 0, & \text{if } i = p. \end{cases}$$

Recall that the Ext groups are defined as the right derived functors of the Hom functor, $\mathrm{Ext}^i_A(\kappa, -) = R^i\mathrm{Hom}_A(\kappa, -)$. That is, $\mathrm{Ext}^i_A(\kappa, M)$ equals the i-th cohomology of the complex $\mathrm{Hom}_A(\kappa, I^\bullet)$, where $0 \to M \to I^\bullet$ is an injective resolution of M.

Proof. This is clearly true if $p = 0$. Thus, let us suppose that it is true for any A-module N and any regular N-sequence with less than p elements. Note now that $(a_2, \dots, a_p)$ is a regular M/a_1M-sequence, so

$$\mathrm{Ext}^{p-1}_A(\kappa, M/a_1M) = \mathrm{Hom}_A(\kappa, M/(a_1, \dots, a_p)M),$$

$$\mathrm{Ext}^i_A(\kappa, M/a_1M) = 0,$$

for $i < p - 1$. Therefore, it suffices to prove that $\mathrm{Ext}^i_A(\kappa, M) \cong \mathrm{Ext}^{i-1}_A(\kappa, M/a_1M)$.

Consider then the exact sequence

$$0 \longrightarrow M \xrightarrow{a_1} M \longrightarrow M/a_1M \longrightarrow 0.$$

If we take the Ext long exact sequence, we have

$$\cdots \to \mathrm{Ext}_A^{i-1}(\kappa, M) \longrightarrow \mathrm{Ext}_A^{i-1}(\kappa, \tfrac{M}{a_1M}) \longrightarrow \mathrm{Ext}_A^{i}(\kappa, M) \xrightarrow{a_1} \mathrm{Ext}_A^{i}(\kappa, M) \to \cdots$$

Since $a_1 \in \mathfrak{m}$, we have that the endomorphism $a_1 : \mathrm{Hom}_A(\kappa, M) \to \mathrm{Hom}_A(\kappa, M)$ is trivial and, by the construction of the derived functors, so it is $a_1 : \mathrm{Ext}_A^i(\kappa, M) \to \mathrm{Ext}_A^i(\kappa, M)$. On the other hand, $\mathrm{Ext}_A^{i-1}(\kappa, M/a_1M) = 0$ for $i < p$, so $\mathrm{Ext}_A^i(\kappa, M) = 0$ for $i < p$.

Finally, for $i = p$ we get an exact sequence

$$0 \longrightarrow \mathrm{Ext}_A^{p-1}(\kappa, M) \longrightarrow \mathrm{Ext}_A^{p-1}(\kappa, \tfrac{M}{a_1M}) \longrightarrow \mathrm{Ext}_A^{p}(\kappa, M) \longrightarrow 0.$$

By the induction hypothesis, we have $\mathrm{Ext}_A^{p-1}(\kappa, M) = \mathrm{Hom}_A(\kappa, M/(a_1, \dots, a_{p-1})M)$. Now, since $(a_1, \dots, a_{p-1})$ can be extended to the longer sequence $(a_1, \dots, a_p)$, we have

$$\mathrm{Hom}_A(\kappa, M/(a_1, \dots, a_{p-1})M) = 0,$$

so $\mathrm{Ext}_A^{p-1}(\kappa, M/a_1M) \cong \mathrm{Ext}_A^{p}(\kappa, M)$. □

We say that a regular M-sequence is *maximal* if it cannot be extended to a longer sequence. The Noetherian condition implies that every sequence can be extended to a maximal one: if not there would be a strictly increasing sequence of ideals

$$(a_1) \subset (a_1, a_2) \subset (a_1, a_2, a_3) \subset \cdots.$$

The previous proposition shows that the length of a maximal sequence only depends on M. We call this number the *depth* of M. We can also define the depth as

$$\mathrm{depth}_A(M) = \min \left\{ n : \mathrm{Ext}_A^n(\kappa, M) \neq 0 \right\}.$$

A nice thing about depth is that it is preserved by finite extensions:

Proposition 1.3. *Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be two Noetherian local rings, and let $\varphi : A \to B$ a local homomorphism (that is, $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$) which makes B into a finitely generated A-module (that is, a* finite *morphism). If M is a finitely generated B-module, then* $\mathrm{depth}_A(M) = \mathrm{depth}_B(M)$.

Recall that for every homomorphism $\varphi : A \to B$ and any module M over B, we can also regard M as an A-module, by defining $ax = \varphi(a)x$, for $a \in A$ and $x \in M$. Geometrically, if we consider the morphism of shemes $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induced by φ and $\tilde{M}$ the quasi-coherent sheaf over $\mathrm{Spec}(B)$ induced by M, we have that the quasi-coherent sheaf over $\mathrm{Spec}(A)$ induced by M, regarded as an A-module, is the direct image sheaf $f_*\tilde{M}$.

Proof. We will write M_A and M_B for M regarded for an A-module and a B-module, respectively. If $(a_1, \dots, a_p)$ is a maximal regular M_A-sequence then, by definition of the A-module structure on M, we have that $(b_1, \dots, b_p)$ is a regular M_B-sequence for $b_i = \varphi(a_i)$.

Now, since the sequence $(a_1, \dots, a_p)$ is maximal, we have that

$$\mathrm{depth}_A(M_A/(a_1, \dots, a_p)M_A) = 0,$$

so $\mathrm{Hom}_A(A/\mathfrak{m}_A, M_A/(a_1, \dots, a_p)M_A) \neq 0$. Therefore there exists a nontrivial module homomorphism $A/\mathfrak{m}_A \to M_A/(a_1, \dots, a_p)M_A$, and so there is some nonzero element $x \in M$ annihilated by $\mathfrak{m}_A$. Let N be the B-module generated by x. This is a nonzero, finitely generated

A-module which is annihilated by $\mathfrak{m}_A$. Hence N has finite length as an A-module, thus also as a B-module. Therefore, there exists some $x \in N$ which is annihilated by $\mathfrak{m}_B$, showing that $\operatorname{depth}_B(M_B/(b_1, \ldots, b_p)M_B) = 0$. We conclude then that $(b_1, \ldots, b_p)$ is a maximal regular M_B-sequence, so $\operatorname{depth}_A(M) = \operatorname{depth}_B(M) = p$. □

Definition 1.4. Let $(A, \mathfrak{m})$ be a Noetherian local ring and M an A-module. We say that M is *Cohen-Macaulay* if $\dim(\operatorname{Supp}(M)) = \operatorname{depth}_A(M)$. Moreover, if $\dim(\operatorname{Supp}(M)) = \operatorname{depth}_A(M) = \dim(A)$, we say that M is a *maximal Cohen-Macaulay* module. We say that A is a *Cohen-Macaulay ring* if A is Cohen-Macaulay as an A-module.

The proposition above shows that being Cohen-Macaulay is preserved by a finite morphism. Moreover, since the dimension of the support is preserved by a finite morphism, we have

Proposition 1.5. *Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be two Noetherian local rings and let $\varphi : A \to B$ a finite local homomorphism. If M is a B-module, then M is (maximal) Cohen-Macaulay as an A-module if and only if it is (maximal) Cohen-Macaulay as a B-module.*

Since we assumed our ring A to be Noetherian, its maximal ideal $\mathfrak{m}$ is generated by a finite number of elements, $\mathfrak{m} = (a_1, \ldots, a_n)$. If we take this n to be minimal then, A is a *regular local ring* precisely if $n = \dim(A)$. This implies that we have an strictly increasing chain of prime ideals

$$(a_1) \subset (a_1, a_2) \subset \cdots \subset (a_1, \ldots, a_n).$$

As a consequence, every a_i is not a zero-divisor of $A/(a_1, \ldots, a_{i-1})$, so $(a_1, \ldots, a_n)$ is a regular A-sequence. Since $\mathfrak{m} = (a_1, \ldots, a_n)$ this is clearly maximal, so $\operatorname{depth}_A(A) = n$. We conclude then

Proposition 1.6. *A regular local ring is Cohen-Macaulay.*

We can generalize the definition of the Cohen-Macaulay property to non-local rings:

Definition 1.7. Let A be a Noetherian ring and M a finitely generated A-module. We say that M is *(maximal) Cohen-Macaulay* if for every prime ideal $\mathfrak{p} \subset A$, the localization $M_\mathfrak{p}$ is (maximal) Cohen-Macaulay over $A_\mathfrak{p}$. We say that A is a *Cohen-Macaulay ring* if A is Cohen-Macaulay as an A-module.

Of course, this also defines the notion of being Cohen-Macaulay for an affine scheme. Since this is a local property, it immediately generalizes to general (Noetherian) schemes:

Definition 1.8. Let X be a Noetherian scheme and $\mathscr{F}$ a coherent sheaf on X. We say that $\mathscr{F}$ is *(maximal) Cohen-Macaulay* if for every point $x \in X$, the stalk $\mathscr{F}_x$ is a (maximal) Cohen-Macaulay module over $\mathscr{O}_{X,x}$. We say that X is a *Cohen-Macaulay scheme* if for every $x \in X$, the ring $\mathscr{O}_{X,x}$ is a Cohen-Macaulay ring.

The result above can be restated now as the fact that a regular scheme is Cohen-Macaulay.

2 The Auslander-Buchsbaum formula

Definition 2.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring, $\kappa = A/\mathfrak{m}$ its residue field, and M a finitely generated A-module. A *minimal free resolution of M* is an exact sequence

$$\cdots \longrightarrow A^{n_i} \xrightarrow{d_i} A^{n_{i-1}} \xrightarrow{d_{i-1}} \cdots \longrightarrow A^{n_1} \xrightarrow{d_1} A^{n_0} \xrightarrow{\varepsilon} M \longrightarrow 0,$$

such that $d_i(A^{n_i}) \subset \mathfrak{m}A^{n_{i-1}}$ for all i and that the induced map $\bar{\varepsilon} : A^{n_0} \otimes \kappa \to M \otimes \kappa$ is an isomorphism.

It is easy to show that every two minimal resolutions are isomorphic as complexes, so we define the *projective dimension* of M as the length $\mathrm{pd}_A(M)$ of any minimal resolution of M.

Example 2.2. Let $(a_1, \dots, a_p)$ a regular A-sequence. We can construct the *Koszul resolution* $K_\bullet$ of $A/(a_1, \dots, a_p)$ as follows:

$$K_0 = A \text{ and } K_k = 0 \text{ if } k > p,$$

and

$$K_k = \bigoplus Ae_{i_1 \dots i_k}$$

is the free A-module of rank $\binom{p}{k}$ with basis $\{e_{i_1 \dots i_k} : 1 \le i_1 < \cdots < i_k \le p\}$. The map $d : K_k \to K_{k-1}$ is defined as

$$d(e_{i_1 \dots i_k}) = \sum_{r=1}^{k} (-1)^{r-1} a_{i_r} e_{i_1 \dots \hat{i}_r \dots i_k}.$$

The modules K_k are free and since, by definition the $a_i \in \mathfrak{m}$, we have that $d(K_k) \subset \mathfrak{m}K_{k-1}$ and

$$A \otimes A/\mathfrak{m} \cong A/(a_1, \dots, a_p) \otimes A/\mathfrak{m}.$$

It remains to check that the complex is exact. In fact, more generally we will prove that the complex $K(M)_\bullet = K_\bullet \otimes_A M$ is exact for M any finitely generated A-module, if $(a_1, \dots, a_p)$ is a regular M-sequence. We will prove it by induction on p. If $p = 1$, we have $K_0 = K_1 = M$ and

$$\begin{aligned} d_1 : K_1 &\longrightarrow K_0 \\ a &\longmapsto a_1 a. \end{aligned}$$

Since a_1 is not a zero-divisor, this homomorphism is injective, so $H_1(K_\bullet) = \ker d_1 = 0$. Suppose now that $p > 1$ and that the Koszul complex $K(i; M)_\bullet$ for the sequence $(a_1, \dots, a_i)$ is exact. We will prove that the Koszul complex $K(i+1; M)_\bullet$ for the sequence $(a_1, \dots, a_{i+1})$ is exact. This follows from the following lemma

Lemma 1. *Let $L_\bullet$ be a complex of A-modules, $a \in A$ and $K_\bullet$ the complex associated to A. For every $k \ge 0$ we have an exact sequence*

$$0 \longrightarrow H_0(K(H_k(L_\bullet))_\bullet) \longrightarrow H_k(K_\bullet \otimes_A L_\bullet) \longrightarrow H_1(K(H_{k-1}(L_\bullet))_\bullet) \longrightarrow 0.$$

Proof. The natural injection $A \to K_\bullet$ gives an embedding of complexes

$$L_\bullet = A \otimes_A L_\bullet \to K_\bullet \otimes L_\bullet.$$

Similarly, the natural projection $K_\bullet \to K_1 = A$ gives a morphism of complexes

$$K_\bullet \otimes_A L_\bullet \to L_{\bullet-1}.$$

We thus get an exact sequence of complexes

$$0 \longrightarrow L_\bullet \longrightarrow K_\bullet \otimes_A L_\bullet \longrightarrow L_{\bullet-1} \longrightarrow 0,$$

and from it, a long exact sequence in homology

$$\cdots \xrightarrow{d} H_k(L_\bullet) \longrightarrow H_k(K_\bullet \otimes_A L_\bullet) \longrightarrow H_k(L_{\bullet-1}) \xrightarrow{d} H_{k-1}(L_\bullet) \longrightarrow \cdots.$$

Now, the map $d : H_k(L_{\bullet-1}) \to H_{k-1}(L_\bullet)$ consists simply on multiplying by a. Therefore, if we consider

$$\operatorname{coker}(a : H_k(L_\bullet) \to H_k(L_\bullet)) = H_0(K(H_k(L_\bullet)_\bullet),$$

$$\ker(a : H_k(L_\bullet) \to H_1(L_\bullet)) = H_0(K(H_k(L_\bullet)_\bullet);$$

we can split the above exact sequence into short exact sequences and we get what we wanted. □

Now, note that $K(i+1; M)_\bullet = K(i; M)_\bullet \otimes K(a_{i+1})_\bullet$, where $K(a_{i+1})_\bullet$ is the Koszul complex associated to a_{i+1}. Therefore, if in the previous proposition we take $L_\bullet = K(i; M)_\bullet$, we have $H_k(K(i+1; M)_\bullet) = 0$.

This then proves that the Koszul complex is exact, and, in particular, it gives a minimal free resolution of $A/(a_1, \dots, a_p)$. Moreover, note that since this is a free resolution, the homology groups $H_k(K(M)_\bullet)$ must be equal to the left derived functors of tensoring by M. These are precisely the Tor functors, so we get

$$\operatorname{Tor}_i^A(A/(a_1, \dots, a_p), M) = H_i(K(M)_\bullet).$$

This example proves that, for any regular A-sequence $(a_1, \dots, a_p)$, the A-module $A/(a_1, \dots, a_p)$ has projective dimension p, and that

$$\operatorname{pd}_A(A/(a_1, \dots, a_p)) \geq \max\left\{i : \operatorname{Tor}_i^A(A/(a_1, \dots, a_p), M) \neq 0\right\}.$$

Let $(A, \mathfrak{m})$ be a Noetherian local ring, $\kappa = A/\mathfrak{m}$ its residue field, and M a finitely generated A-module. Note that for every minimal free resolution $A^{n_\bullet} \to M$, by definition of the derived functors, we have that $\operatorname{Tor}_i^A(M, \kappa) = H_i(A^{n_\bullet} \otimes \kappa)$. Moreover, from the definition of minimal resolution, $d_i(A^{n_i}) \subset \mathfrak{m}A^{n_{i-1}}$, so $H_i(A^{n_\bullet} \otimes \kappa) = A^{n_i} \otimes \kappa$, and the dimension of this as a κ-vector space is equal to n_i. Therefore,

$$\operatorname{pd}_A(M) = \max\{i : n_i \neq 0\} = \max\left\{i : \operatorname{Tor}_i^A(M, \kappa) \neq 0\right\}.$$

Suppose now that $(A, \mathfrak{m})$ is a regular local ring. We showed that, if $\mathfrak{m} = (a_1, \dots, a_n)$, then $(a_1, \dots, a_n)$ is a regular A-sequence, and we have that $\operatorname{pd}_A(\kappa) \geq \max\left\{i : \operatorname{Tor}_i^A(\kappa, M) \neq 0\right\}$. Now, since $\operatorname{Tor}_i^A(M, \kappa) = \operatorname{Tor}_i^A(\kappa, M)$, the projective dimension of any finitely generated A-module M is bounded by

$$\operatorname{pd}_A(M) \leq \operatorname{pd}_A(\kappa) = n.$$

To sum up, if A is a regular local ring then every finitely generated A-module has finite projective dimension.

We can now conclude some interesting results from the following formula:

Theorem 2.3 (Auslander-Buchsbaum formula)**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated A-module. Suppose that* $\operatorname{pd}_A(M)$ *is finite; then*

$$\operatorname{pd}_A(M) + \operatorname{depth}_A(M) = \operatorname{depth}_A(A).$$

Proof. Let us call $h = \operatorname{pd}_A(M)$; we will work by induction on h. If $h = 0$, then M is a free A-module, so the assertion is trivial. If $h = 1$, let

$$0 \longrightarrow A^m \xrightarrow{\varphi} A^n \xrightarrow{\varepsilon} M \longrightarrow 0$$

be a minimal resolution of M. We can write φ as an $m \times n$ matrix with entries in $\mathfrak{m}$. From this exact sequence we obtain the Ext long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_A(k, A^m) \xrightarrow{\varphi_*} \mathrm{Ext}^i_A(k, A^n) \xrightarrow{\varepsilon_*} \mathrm{Ext}^i_A(k, M) \longrightarrow \cdots,$$

and, since we can write $\mathrm{Ext}^i_A(k, A^m) = \mathrm{Ext}^i_A(k, A)^m$ and $\mathrm{Ext}^i_A(k, A^n) = \mathrm{Ext}^i_A(k, A)^n$, we can express φ_* by the same matrix as φ. Since the entries of this matrix are elements of $\mathfrak{m}$, they annihilate $\mathrm{Ext}^i_A(k, A)$, so $\varphi_* = 0$, and for every i we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^i_A(k, A)^n \longrightarrow \mathrm{Ext}^i_A(k, M) \longrightarrow \mathrm{Ext}^{i+1}_A(k, A)^m \longrightarrow 0.$$

Since $\mathrm{depth}(M) = \min\{i : \mathrm{Ext}^i_A(k, M) \neq 0\}$, we have $\mathrm{depth}(M) = \mathrm{depth}(A) - 1$ and the theorem holds if $h = 1$. If $h > 1$, for any exact sequence $0 \to M' \to A^n \to M \to 0$ we have $\mathrm{pd}_A(M') = h - 1$ and we finish by an easy induction. □

Corollary 2.4. *Let $(A, \mathfrak{m})$ be a Noetherian regular local ring. Then M is a maximal Cohen-Macaulay A-module if and only if it is free.*

Proof. Since A is regular, $\mathrm{pd}_A(M)$ is finite and we can apply the Auslander-Buchsbaum formula. Moreover, since A is regular, in particular it is Cohen-Macaulay, $\mathrm{depth}_A(A) = \dim(A)$. Therefore, if M is maximal Cohen-Macaulay, then $\mathrm{depth}_A(M) = \dim(A) = \mathrm{depth}_A(A)$ so, by the Auslander-Buchsbaum formula, $\mathrm{pd}_A(M) = 0$, which implies that M is free. □

In the geometric picture this can be restated as the fact that every maximal Cohen-Macaulay sheaf over a regular scheme is locally free.

Combining this fact with Proposition 1.5, we have

Proposition 2.5. *Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be two Noetherian local rings and let $\varphi : A \to B$ a finite local homomorphism. Moreover, suppose that A is regular. If M is a finitely generated B-module, then M is maximal Cohen-Macaulay as a B-module if and only if it is free as an A-module.*

3 Quasi-finite morphisms and completions

We would like to generalize this result to rings that are not local. The main problem for doing this is that, if $A \to B$ is a finite morphism, then its localization $A_{\varphi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$ might not be finite. We can see this with an example:

Example 3.1. Consider k an algebraically closed field and the homomorphism

$$\begin{aligned} \varphi : k[X] &\longrightarrow k[X] \\ p(X) &\longmapsto p(X^2 - 1). \end{aligned}$$

This is clearly a finite morphism of degree 2. We can consider now the maximal ideal $\mathfrak{m} = (X - 1) \subset k[X]$. It inverse image is

$$\varphi^{-1}(\mathfrak{m}) = \mathfrak{m} \cap \varphi(k[X]) = \mathfrak{m} \cap k[X^2 - 1] = (X - 1) \cap k[X^2 - 1].$$

However, $(X + 1) \cap k[X^2 - 1] = \mathfrak{m}$. Indeed, a polynomial on $k[X^2 - 1]$ can be written as

$$p(X^2-1) = a_n(X^2-1)^n + \cdots + a_1(X^2-1) + a_0 = a_n(X+1)^n(X-1)^n + \cdots + a_1(X+1)(X-1) + a_0.$$

Therefore, $X + 1$ divides $p(X^2 - 1)$ if and only if $X - 1$ does.

Let us prove now that the localization $k[X]_{\varphi^{-1}(\mathfrak{m})} \to k[X]_{\mathfrak{m}}$ is not finite. In particular, we will prove that $k[X^2-1]_{(X+1)\cap k[X^2-1]} \hookrightarrow k[X]_{(X-1)}$ is not an integral extension. Indeed, suppose that $\frac{1}{X+1} \in k[X]_{(X-1)}$ is integral over $k[X^2-1]_{(X+1)\cap k[X^2-1]}$. Then we can find elements $a_i \in k[X^2-1]$, and $s_i \in k[X^2-1] \setminus ((X+1) \cap k[X^2-1])$, for $i = 1, \dots, n$ and $n \in \mathbb{N}$, such that

$$\left(\frac{1}{X+1}\right)^n + \frac{a_1}{s_1}\left(\frac{1}{X+1}\right)^{n-1} + \cdots + \frac{a_{n-1}}{s_{n-1}}\frac{1}{X+1} + \frac{a_n}{s_n} = 0.$$

Clearing denominators we get

$$s_1 \cdots s_n + a_1 s_2 \cdots s_n (X+1) + \cdots + b_{n-1} s_1 \cdots s_{n-2} s_n (X+1)^{n-1} + a_n s_1 \cdots s_{n-1}(X+1)^n = 0.$$

This implies that there is some polynomial $q(X^2-1) \in k[X^2-1]$ such that

$$s_1 \cdots s_n = (X+1) q(X^2-1).$$

In particular, $(X+1)$ divides $s_1 \cdots s_n$ in $k[X]$, so $s_1 \cdots s_n \in (X+1) \cap k[X^2-1]$. But this is a prime ideal, so we have arrived at a contradiction with the definition of the s_i.

As we said above, the previous example shows how the finiteness of a morphism need not be preserved after localization. However, a slightly weaker condition will be preserved: quasi-finiteness.

Definition 3.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring. We say that an A-module M is *quasi-finite* if $M/\mathfrak{m}M$ is finite-dimensional as a κ-vector space, where $\kappa = A/\mathfrak{m}$ is the residue field. In particular, we say that a local homomorphism of local rings $\varphi : A \to B$ is *quasi-finite* if B is a quasi-finite A-module.

Remark. If $\varphi : A \to B$ is a finite homomorphism of (non-local) rings, then, for every prime ideal $\mathfrak{p} \subset B$, if we call $\mathfrak{q} = \varphi^{-1}(\mathfrak{p})$, we have a local homomorphism $A_{\mathfrak{q}} \to B_{\mathfrak{p}}$. Now, this morphism will be quasi-finite if $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is finite-dimensional as a vector space over $\kappa(\mathfrak{q}) = A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}$. But note now that we can write

$$\frac{B_{\mathfrak{p}}}{\mathfrak{p}B_{\mathfrak{p}}} = B \otimes_A \kappa(\mathfrak{q})$$

so, since B is a finite A-algebra, $B \otimes_A \kappa(\mathfrak{q})$ has finite dimension over $\kappa(\mathfrak{q})$. We conclude then that if $\varphi : A \to B$ is a finite homomorphism then it is quasi-finite in every localization.

Geometrically, its easy to check that if we consider the morphism of schemes $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ induced by φ, then, for every point $x \in \operatorname{Spec}(A)$, the fibre of x is

$$f^{-1}(x) = \operatorname{Spec}(B \otimes_A \kappa(x)).$$

Since this is the spectrum of a finite-dimensional $\kappa(x)$-algebra, it consists on a finite collection of isolated points. This provides motivation for the following definition:

Definition 3.3. We say that a morphism of Noetherian schemes $f : Y \to X$ is *quasi-finite* if, for every point $x \in X$, the fibre $f^{-1}(x)$ is a finite set.

The last step consists in recovering finiteness from quasi-finite morphisms. We can achieve this by going to the completions. Recall that if $(A, \mathfrak{m})$ is a Noetherian local ring, then we can endow any finitely generated A-module M with the topology induced by the filtration

$$M \supseteq \mathfrak{m}M \supseteq \mathfrak{m}^2 M \supseteq \cdots \supseteq \mathfrak{m}^n M \supseteq \cdots.$$

This is called the $\mathfrak{m}$-*adic topology* on M. We define the *completion* $\hat{M}$ of M as the completion of M with respect to this topology, that is

$$\hat{M} = \varprojlim_n \frac{M}{\mathfrak{m}^n M}.$$

The key fact now is the following.

Proposition 3.4. *Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be two Noetherian local rings and $\varphi : A \to B$ a quasi-finite local homomorphism. Then:*

1. *For any quasi-finite B-module M, the $\mathfrak{m}_B$-adic topology and the $\varphi(\mathfrak{m}_A)B$-adic topology on M coincide.*

2. *The induced homomorphism in the completions $\hat{\varphi} : \hat{A} \to \hat{B}$ is finite.*

Proof.
1. First, we are going to show that $\mathfrak{m}_B = \sqrt{\varphi(\mathfrak{m}_A)B}$. That is, there exists some $n \in \mathbb{N}$ such that $\mathfrak{m}_B^n \subset \varphi(\mathfrak{m}_A)B$. Since B is a quasi-finite A-module, $B/\varphi(\mathfrak{m}_A)B$ is a finite-dimensional $A/\mathfrak{m}_A$-vector space. Therefore, the chain of B-modules

$$\varphi(\mathfrak{m}_A)B \subset \cdots \subset \mathfrak{m}_B^{n+1} + \varphi(\mathfrak{m}_A)B \subset \mathfrak{m}_B^n + \varphi(\mathfrak{m}_A)B \subset \cdots \subset \mathfrak{m}_B + \varphi(\mathfrak{m}_A)B \subset B,$$

has to be finite. Thus, there exists some $n \geq 1$ such that $\mathfrak{m}_B^{n+1} + \varphi(\mathfrak{m}_A)B = \mathfrak{m}_B^n + \varphi(\mathfrak{m}_A)B$, so

$$\mathfrak{m}_B(\mathfrak{m}_B^n + \varphi(\mathfrak{m}_A)B) + \varphi(\mathfrak{m}_A)B = \mathfrak{m}_B^n + \varphi(\mathfrak{m}_A)B.$$

Therefore, by the Nakayama lemma, $\varphi(\mathfrak{m}_A)B = \mathfrak{m}_B^n + \varphi(\mathfrak{m}_A)B$. Thus, $\mathfrak{m}_B^n \subset \varphi(\mathfrak{m}_A)B$.

The two topologies coincide when for any $n \in \mathbb{N}$ there is some $m \in \mathbb{N}$ such that $\mathfrak{m}_B^m \subset \varphi(\mathfrak{m}_A)^n B$ and some m' such that $\varphi(\mathfrak{m}_A)^{m'} B \subset \mathfrak{m}_B^n$. Since φ is a local homomorphism, $\varphi(\mathfrak{m}_A)B \subset \mathfrak{m}_B$, so $\varphi(\mathfrak{m}_A)^n B \subset \mathfrak{m}_B^n$ for every $n \in \mathbb{N}$. On the other hand, since we showed that $\mathfrak{m}_B = \sqrt{\varphi(\mathfrak{m}_A)B}$, there exists some $n_0 \in \mathbb{N}$ such that $\mathfrak{m}_B^{n_0} \subset \varphi(\mathfrak{m}_A)B$. Therefore, $\mathfrak{m}_B^{n_0 n} \subset \varphi(\mathfrak{m}_A)^n B$.

2. Since B is a quasi-finite A-module, there exists a surjection $\psi' : \kappa^m \to B/\varphi(\mathfrak{m}_A)B$, where $\kappa = A/\mathfrak{m}_A$ is the residue field of A, for some $m \in \mathbb{N}$. That is, we have a surjection

$$\psi' : A^m/\mathfrak{m}_A A^m \to B/\varphi(\mathfrak{m}_A)B.$$

We can lift this to an A-module homomorphism $\psi : A^m \to B$. By Nakayama's lemma, the induced map $A^m/\mathfrak{m}_A^n A^m \to B/\varphi(\mathfrak{m}_A)^n B$ is surjective for each $n > 0$. Now, set

$$K_n = \{x \in A^m : \psi(x) \in \varphi(\mathfrak{m}_A)^n B\}.$$

We get short exact sequences

$$0 \longrightarrow K_n/\mathfrak{m}_A^n A^m \longrightarrow A^m/\mathfrak{m}_A^n A^m \longrightarrow B/\varphi(\mathfrak{m}_A)^n B \longrightarrow 0.$$

Thus, for every $n > 0$, we get the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{n+1}/\mathfrak{m}_A^{n+1}A^m & \longrightarrow & A^m/\mathfrak{m}_A^{n+1}A^m & \longrightarrow & B/\varphi(\mathfrak{m}_A)^{n+1}B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow{\scriptstyle \beta_{n+1}} & & \downarrow{\scriptstyle \gamma_{n+1}} & & \\
0 & \longrightarrow & K_n/\mathfrak{m}_A^n A^m & \longrightarrow & A^m/\mathfrak{m}_A^n A^m & \longrightarrow & B/\varphi(\mathfrak{m}_A)^n B & \longrightarrow & 0.
\end{array}$$

We claim that $\alpha_{n+1} : K_{n+1}/\mathfrak{m}_A^{n+1}A^m \to K_n/\mathfrak{m}_A^n A^m$ is surjective. Namely, if $x \in K_n$, write $\psi(x) = \sum_j \varphi(y_j)b_j$, with $y_j \in \mathfrak{m}_A^n$, $b_j \in B$. Since ψ' is surjective, we can write $b_j = \psi(z_j) + \sum_k \varphi(y_{jk})b_{jk}$, with $z_j \in A^m$, $y_{jk} \in \mathfrak{m}_A$ and $b_{jk} \in B$. Hence,

$$\psi(x - \sum_j y_j z_j) = \sum_{j,k} \varphi(y_j y_{jk})b_{jk} \in \varphi(\mathfrak{m}_A)^{n+1}B.$$

This means that $x' = x - \sum_j y_j z_j \in K_{n+1}$ maps to x, which proves the claim.

Let us define now,

$$F = \prod_{n=1}^{\infty} K_n/\mathfrak{m}_A^n A^m, \quad G = \prod_{n=1}^{\infty} A^m/\mathfrak{m}_A^n A^m, \quad H = \prod_{n=1}^{\infty} B/\varphi(\mathfrak{m}_B)^n B,$$

and

$$\begin{aligned} d^F : F &\longrightarrow F \\ (f_n)_n &\longmapsto (f_n - \alpha_{n+1}(f_{n+1}))_n. \end{aligned}$$

Define similarly d^G and d^H and note that

$$\varprojlim_n K_n/\mathfrak{m}_A^n A^m = \ker d^F, \quad \hat{A}^m = \varprojlim_n A^m/\mathfrak{m}_A^n A^m = \ker d^G, \quad \hat{B} = \varprojlim_n B/\varphi(\mathfrak{m}_A)^n B = \ker d^H.$$

The diagram above induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^F} & & \downarrow{\scriptstyle d^G} & & \downarrow{\scriptstyle d^H} & & \\ 0 & \longrightarrow & F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0. \end{array}$$

The snake lemma yields now an exact sequence

$$0 \to \ker d^F \to \ker d^G \to \ker d^H \to \operatorname{coker} d^F \to \operatorname{coker} d^G \to \operatorname{coker} d^H \to 0.$$

Finally, since the α_n are surjective, we can solve inductively the equations

$$x_n - \alpha_{n+1}(x_{n+1}) = f_n,$$

for $x_n \in K_n/\mathfrak{m}_A^n A^m$, given $f_n \in K_n/\mathfrak{m}_A^n A^m$, thus proving that d^F is surjective. Therefore, $\operatorname{coker} d^F = 0$ and we get a short exact sequence

$$0 \longrightarrow \ker d^F \longrightarrow \ker d^G = \hat{A}^m \longrightarrow \ker d^H = \hat{B} \longrightarrow 0,$$

in particular, we see that $\hat{A}^m \to \hat{B}$ is surjective, showing that $\hat{B}$ is a finitely generated A-module. □

The only other remark that we need to make is that, since $\hat{A} \otimes_A \operatorname{Ext}^i_A(\kappa, M)$ is isomorphic to $\operatorname{Ext}^i_{\hat{A}}(\kappa, \hat{M})$, depth is preserved under completion.

Now we are in position to generalize Proposition 2.5 to the non-local setting.

Theorem 3.5. *Let A and B be two Noetherian rings and let $\varphi : A \to B$ be a finite homomorphism. Moreover, suppose that A is regular. If M is a finitely generated B-module, then M is maximal Cohen-Macaulay as a B-module if and only if it is locally free as an A-module.*

Proof. Let $\mathfrak{p} \subset B$ be a prime ideal and $\mathfrak{q} = \varphi^{-1}(\mathfrak{p})$. We have that $A_\mathfrak{q} \to B_\mathfrak{p}$ is a local homomorphism and it is quasi-finite since $\varphi : A \to B$ is finite. Therefore, the completion $\hat{\varphi} : \hat{A}_\mathfrak{q} \to \hat{B}_\mathfrak{p}$ is finite. Moreover, the completion $\hat{M}_\mathfrak{p}$ is well defined since the topologies endowed on it by $B_\mathfrak{p}$ and $A_\mathfrak{q}$ are equal. We can now apply Proposition 2.5 to obtain that $\hat{M}_\mathfrak{p}$ is maximal Cohen-Macaulay as a $\hat{B}_\mathfrak{p}$-module if and only if it is free as an $\hat{A}_\mathfrak{q}$-module. Now, since the depth and the dimension are preserved by completion, we have that $M_\mathfrak{p}$ is maximal Cohen-Macaulay as a $B_\mathfrak{p}$-module if and only if it is free as an $A_\mathfrak{q}$-module. □

We can understand this result geometrically in the following way. Suppose that X and Y are Noetherian schemes and $f : Y \to X$ is a finite morphism. In general, since f is an affine morphism, the direct image f_* gives an equivalence of categories,

$$\left\{\text{Quasi-coherent sheaves over } Y\right\} \xrightarrow{f_*} \left\{\begin{matrix}\text{Quasi-coherent sheaves over } X \\ \text{with } f_*\mathcal{O}_Y\text{-module structure}\end{matrix}\right\}.$$

This is just a fancy way of saying that for any map of rings $A \to B$, a module over B naturally inherits an A-module structure, as it was explained above. What the theorem is telling us is that, if X is a regular scheme, this correspondence descends to an equivalence of categories

$$\left\{\text{Maximal Cohen-Macaulay sheaves over } Y\right\} \xrightarrow{f_*} \left\{\begin{matrix}\text{Vector bundles over } X \\ \text{with } f_*\mathcal{O}_Y\text{-module structure}\end{matrix}\right\}.$$