The Hitchin fibration: Analogues and generalizations

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- The corresponding line bundle on Y_a is given by the eigenvectors.

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Equivalently, we get an isomorphism

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Theorem (Donagi-Gaitsgory)

The general fibre $h_{G,L} : \mathcal{M}_{G,L} \longrightarrow \mathcal{A}_{G,L}$ is a certain moduli space of T-bundles with extra structure on the corresponding cameral curve.

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- We are going to study some examples arising naturally from the theory of Higgs bundles.

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- When X is a smooth variety of dimension d and $V = \Omega_X^1$, this is the usual higher-dimensional Hitchin fibration as defined by Simpson.

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$$\chi_{n,d}(x, [x_1, \ldots, x_n]) = x^n + p_1(x_1, \ldots, x_n)x^{n-1} + \cdots + p_n(x_1, \ldots, x_n),$$

for p_i the i-th elementary symmetric polynomial in n variables.

- Spectral covers constructed from

$$\begin{array}{ccc} Y_a & \longrightarrow & V \times_{\mathrm{GL}_d} \chi^{-1}_{n,d}(0) \\ \downarrow & & \downarrow{\scriptstyle V \times_{\mathrm{GL}_d} \mathrm{pr}_2} \\ X & \xrightarrow{a} & V \times_{\mathrm{GL}_d} \mathrm{Sym}^n \mathbb{A}^d_k. \end{array}$$

- These spectral covers are not flat in general! But for $\dim X \leq 2$ there are flat modifications [Chen-Ngô]. In [G–García-Prada–Narasimhan] we study them in detail for some cases over $\dim X = 1$.

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for p_i the i-th elementary symmetric polynomial in n variables.

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Example 3: First approach. Meromorphic sections

- $\mathcal{M} = \{(E, \varphi) : E \longrightarrow X \; G\text{-bundle}, \varphi \in H^0(X', E \times_{\mathrm{Ad}} G) \text{ for } X' = X \setminus \text{finite subset}\}.$

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- Construct a Hitchin fibration from the sequence

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- Construct some monoid M_λ as the pullback of this map.
- Recover the multiplicative Hitchin fibration from the sequence

$$[\mathrm{Ad}(G)\backslash M_\lambda/\mathbb{G}^n_m] \longrightarrow [C_{M_\lambda}/\mathbb{G}^n_m] \longrightarrow [\mathbb{A}^n_k/\mathbb{G}^n_m] \longrightarrow \mathbb{B}\mathbb{G}^n_m.$$

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- Possible generalization to spherical varieties.

Questions?