Multiplicative Higgs bundles

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Part I

Higgs bundles and the Hitchin system

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Example

Take $G = \mathrm{GL}_n$ and $\mathfrak{t}$ the diagonal matrices, then $p_1, \dots, p_n$ are defined by the characteristic polynomial

$$\det(t - A) = t^n + \sum_{i=1}^{n} p_i(A) t^{n-i}.$$

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- "Miracle":

$$\dim \mathcal{B} = \frac{1}{2} \dim \mathbf{Higgs}_G.$$

The Hitchin system

- The Hitchin map is defined as

$$\begin{aligned} h : \mathbf{Higgs}_G &\longrightarrow \mathcal{B} \\ [E, \varphi] &\longmapsto (p_1(\varphi), \cdots, p_k(\varphi)). \end{aligned}$$

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Theorem (Hitchin, 1987)

The Hitchin map h is an algebraically completely integrable system (i.e. the generic fibre is a Lagrangian abelian variety).

Part II

The multiplicative Hitchin system

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- This suggests the existence of a multiplicative Hitchin system.

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Here, φ meromorphic means that it is defined over $X \setminus D$, for $D = \{x_1, \ldots, x_n\}$ a finite subset of points, called the singular points of φ.

Meromorphic data (1)

- Take (E, φ) a multiplicative G-Higgs bundle and x a singular point of it.

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- $Z(\mathcal{O}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{D}, Z)$, $Z(K) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{D}^*, Z)$ — positive formal loop space and formal loop space, for any $\mathbb{C}$-space Z.

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- The meromorphic datum of φ at x is the double coset

$$[\varphi|_{\mathbb{D}^*}] \in G(\mathcal{O})\backslash G(K)/G(\mathcal{O}).$$

Meromorphic data (2)

Theorem (Iwahori–Matsumoto, 1965)

Let $B \subset G$ be a Borel subgroup containing T and consider $\Lambda_G^+ \subset \mathrm{Hom}(\mathbb{G}_m, T)$ the corresponding subset of dominant cocharacters.

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Thus, the meromorphic data of a multiplicative Higgs bundle (E, φ) with singular locus $D = \{x_1, \ldots, x_n\}$ are given by

$$\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\Lambda_G^+)^n.$$

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- For any singular locus $D = \{x_1, \ldots, x_n\}$ and meromorphic data $\vec{\lambda} \in (\Lambda_G^+)^n$ Hurtubise–Markman (2002) construct some variety $B_{D,\vec{\lambda}} \to X$ so that

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- The multiplicative Hitchin base is the space $\mathcal{B}_{D,\vec{\lambda}}$ of sections of $B_{D,\vec{\lambda}}$.

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(In fact, Elliot–Pestun (2019) showed that, in that case, $\mathbf{mHiggs}_{G,D,\vec{\lambda}}$ is hyperkähler).

Part III

Further directions

Related topics

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- Elliot–Pestun (2019) suggest the existence of a "multiplicative geometric Langlands correspondence".

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- There is an isomorphism of G-varieties

$$\begin{aligned} G/G^\theta &\longrightarrow G *_\theta e \\ gG^\theta &\longmapsto g *_\theta e = g\theta(g)^{-1}. \end{aligned}$$

Involutions on the moduli space. The fixed points

Theorem

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1. E *reduces to some* $G^{\theta'}$ *bundle* E_s*, for some* $s \in G$ *such that* $s\theta(s) \in Z(G)$ *and* $\theta' = s\theta(-)s^{-1}$.

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2. - *If $\epsilon = 1$, then φ defines a section of $E_s \times_{G^{\theta'}} G^{\theta'}$.*
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- Recall that $G *_{\theta'} t = (G *_{\theta''} e)t$, for $\theta'' = t\theta'(-)t^{-1}$.
- In particular, for $\epsilon = -1$ one obtains pairs of the form (E, φ), for E a $G^{\theta'}$-bundle and φ a section of $G/G^{\theta'}$.

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Involutions on the moduli space. The fixed points

Theorem

Let $\theta \in \mathrm{Aut}_2(G)$ and (E, φ) a simple multiplicative G-Higgs bundle. Suppose that (E, φ) is isomorphic to $(\theta(E), \theta(\varphi)^\epsilon)$, for $\epsilon \in \{+1, -1\}$. Then,

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- The meromorphic data of the fixed points of ι_θ^- lie in the subset of dominant weights $\Lambda_A \cap \Lambda_G^+$, for A a stable θ'-anisotropic torus, and some θ' in the same clique as θ.

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- Hitchin map? For M symmetric, there is a "multiplicative Chevalley–Konstant–Rallis" (Richardson, 1982).

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Thank you