Higgs bundles twisted by a vector bundle

- By: Guillermo Gallego

(Universidad Complutense de Madrid)

4th of February 2022 @ Albert-Ludwigs Universität Freiburg

Higgs bundles twisted by a vector bundle

[Based on arXiv: 2105.05543,
joint work w/ Oscar García-Prada
M.S. Narasimhan]

§1. Higgs bundles

- X smooth projective curve / $\mathbb{C}$.
  K_X canonical line bundle

Def: (Hitchin, 1987)

A Higgs bundle on X is a pair (E, φ)

- $E \to X$ vector bundle
- $\varphi : E \to E \otimes K_X$. (idea: K_X-twisted endomorphism).

- X smooth proj. variety $/\mathbb{C}$ of dim. d.

Ω^1_X cotangent bundle.

<u>Def.</u> (Simpson, 1988)

A <u>Higgs bundle</u> on X is a pair (E,φ)

- $E \to X$ vector bundle
- $\varphi : E \to E \otimes \Omega^1_X$.
- (Commuting condition) $\varphi \wedge \varphi = 0$.

(Integrability) $\in H^0(\mathrm{End}\, E \otimes \Lambda^2 V)$

<u>Idea:</u> Pick $\{\omega_1, \dots, \omega_d\}$ local frame,

$$\varphi = \sum_{i=1}^{d} \varphi_i \otimes \omega_i , \quad \varphi_i \in H^0(\mathrm{End}\, E)$$

$$\varphi \wedge \varphi = \sum_{i<j} [\varphi_i, \varphi_j] \otimes \omega_i \wedge \omega_j .$$

- <u>Generalization</u>: Change Ω^1_X by any vector bundle V of arbitrary rank r.

Def. Let $V \longrightarrow X$ be a vector bundle of rank r.

A <u>V-twisted Higgs bundle</u> is a pair (E, φ)

- $E \to X$ a vector bundle
- $\varphi : E \longrightarrow E \otimes V$
- $\varphi \wedge \varphi = 0$.

<u>Remarks</u>

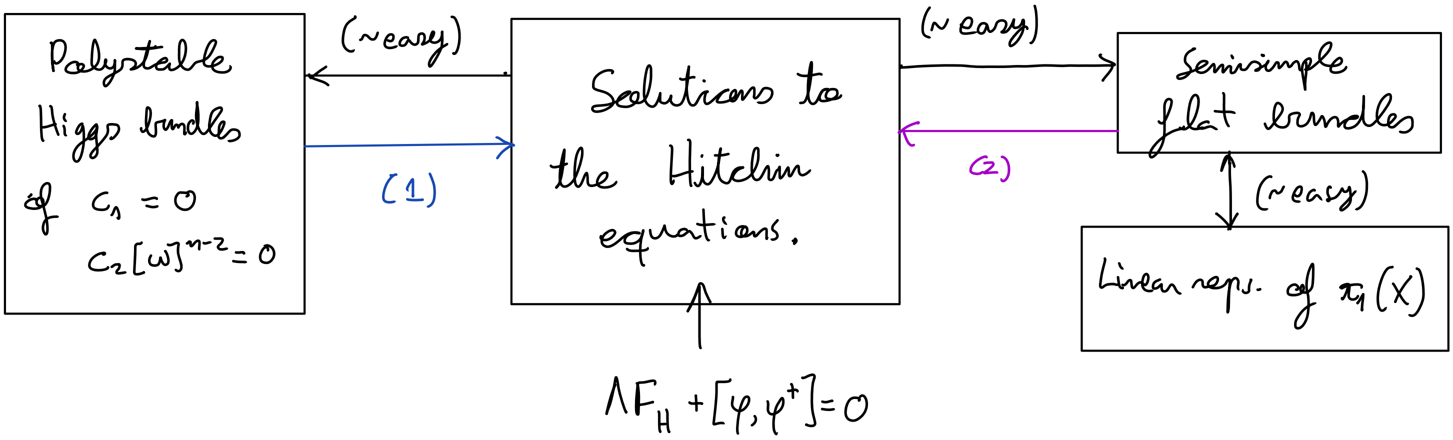
- When $\dim_{\mathbb{C}} X = 1$, these objects are an interesting "halfway" between Higgs bundles on curves and on higher dimension.

- These objects have appeared sporadically in the literature and are relevant in some Physics problems.

- (Semi-) stability conditions can be defined for all these objects, and moduli spaces can be constructed, following the general lines of [Simpson: Moduli of rep. ... I].

§2. Why are Higgs bundles famous?

- **Reason 1: The nonabelian Hodge correspondence**



(1) Hitchin - Simpson:

- Existence of Hermitian-Einstein metrics on Higgs bundles.
- Generalization of Donaldson-Uhlenbeck-Yau (Narasimhan-Seshadri in $\dim X = 1$).
- Can be seen as a "Hitchin-Kobayashi correspondence"
  ↳ Relate Kähler quot. with GIT quot. (∞-dim. analogue of Kempf-Ness).
- Can be generalized to the "V-twisted" case.

(2) Donaldson – Corlette :

- Existence of harmonic metrics on flat bundles.
- Only works for "usual" Higgs bundles.

- Reason 2: The hyperkähler structure

The moduli space of Higgs bundles $\mathcal{M}$ can be endowed with 2 different nonisomorphic complex structures:

I ⇝ coming from Higgs bundles

J ⇝ coming from the character variety $\left(\mathrm{Hom}(\pi_1(X), GL_n)\right)$.

- $(I, J, K = I \cdot J)$ have quaternionic relations

- Moreover, in dimension 1 the character variety is naturally symplectic.

Hyperkähler manifold

- Reason 3 : The Hitchin map

  - Hitchin, 1987 : for Higgs bundles over $\dim X = 1$.
  - Beauville - Narasimhan - Ramanan : for L-twisted Higgs bundles over $\dim X = 1$, L a line bundle.
  - Simpson, 1991 : defines it over $\dim X > 1$.
  - Chen - Ngô, 2020 : study the fibres for $\dim X = 2$.
  - G - García Prada - Narasimhan, 2021 : study the fibres for V-twisted Higgs bundles and $\dim X = 1$, $\mathrm{rk}\, V = 2$.

§ 3. The Hitchin map

- X smooth projective variety $/ \mathbb{C}$
- $V \longrightarrow X$ vector bundle of rank r.

$\mathcal{M}_{n,V} \equiv$ Moduli space of V-twisted Higgs bundles of rank n.

- Hitchin map:

$$h_{n,V} : \mathcal{M}_{n,V} \longrightarrow \bigoplus_{i=1}^{n} H^0(X, S^i V)$$

$$(E,\varphi) \longmapsto (\sigma_1(\varphi), \dots, \sigma_n(\varphi)),$$

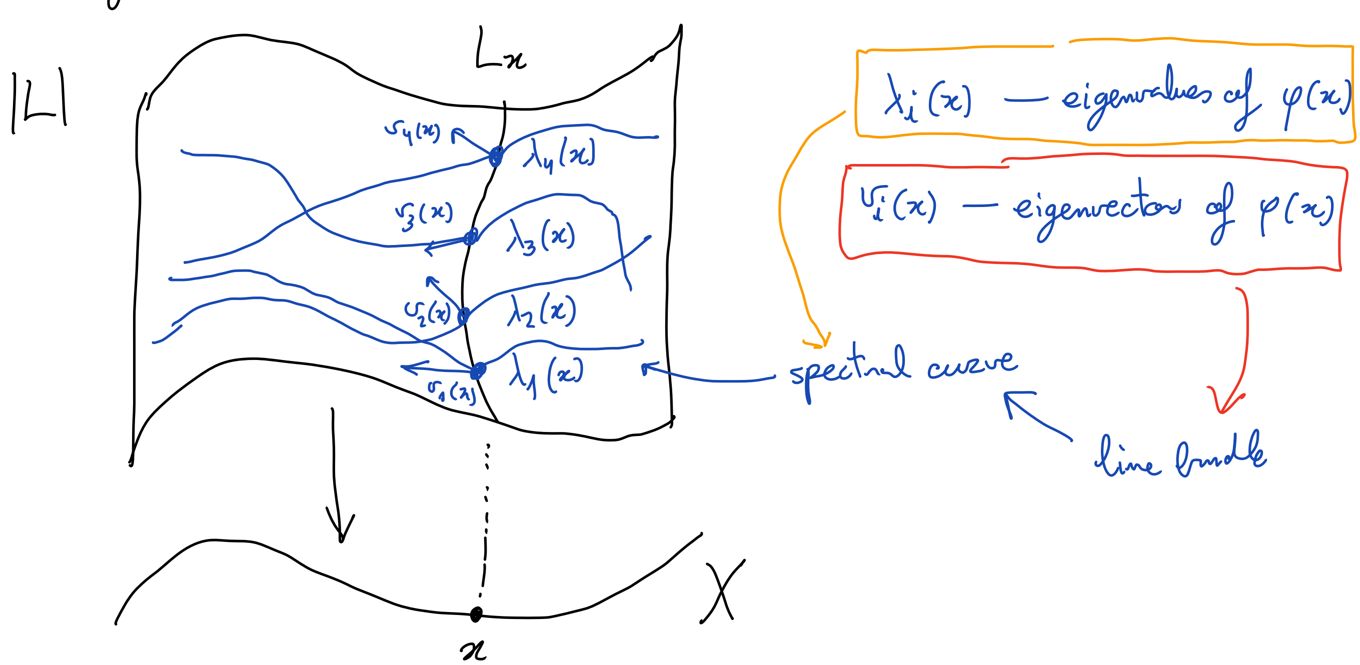
where σ_i are given by

$$P_\varphi(T) = \det(T-\varphi) = T^n + \sum_{i=1}^{n} \sigma_i(\varphi) T^{n-i}.$$

- Hitchin base: $\operatorname{im} h_{n,V}$.

- <u>Problem</u>: $\boxed{\text{Study the fibres of } h_{n,V}}$

<u>Idea</u>: If $V=L$ is a line bundle



§ 4. Universal spectral data [Chen-Ngo]

E $\quad\mathbb{C}$-vector space of dim. n.

$\varphi_1, \dots, \varphi_r \in \operatorname{End} E \quad$ s.t. $\quad [\varphi_i, \varphi_j] = 0$.

$$A := \mathbb{C}[\varphi_1, \dots, \varphi_r]$$

$$I \hookrightarrow \mathbb{C}[x_1, \dots, x_r] \xrightarrow{ev} A \subset \operatorname{End} E \rightsquigarrow$$ $\mathbb{C}[x_1, \dots, x_r]$-module str. on E

primary decomposition

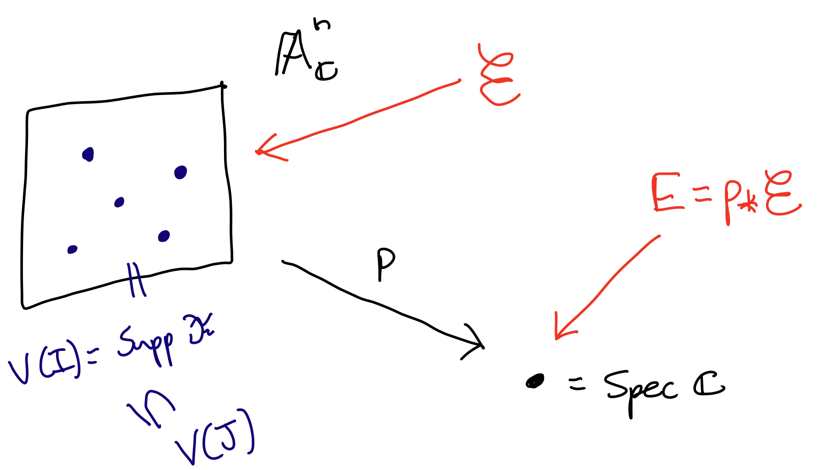
$$I = m_1^{\alpha_1} \cdots m_s^{\alpha_s} \supseteq m_1^{n_1} \cdots m_s^{n_s} = J$$

Nakayama

[Generalized Cayley-Hamilton]

$$E = \bigoplus_{i=1}^{s} E_i \quad , \quad n_i = \dim E_i$$

Geometrically:



E $\quad\mathbb{C}$-vector space of dim. n.

$\varphi_1, \dots, \varphi_r \in \operatorname{End} E \quad$ s.t. $\quad [\varphi_i, \varphi_j] = 0$.

$$A := \mathbb{C}[\varphi_1, \dots, \varphi_r]$$

$$I \hookrightarrow \mathbb{C}[x_1, \dots, x_r] \xrightarrow{ev} A \subset \operatorname{End} E \rightsquigarrow$$ $\mathbb{C}[x_1, \dots, x_r]$-module str. on E

primary decomposition

$$I = m_1^{\alpha_1} \cdots m_s^{\alpha_s} \supseteq m_1^{n_1} \cdots m_s^{n_s} = J$$

Nakayama

[Generalized Cayley-Hamilton]

$$E = \bigoplus_{i=1}^{s} E_i \quad , \quad n_i = \dim E_i$$

- Spectral datum of A: $\quad sd(A)=\sum_{i=1}^{s} n_i[x_i] \in S^m\mathbb{A}^n_{\mathbb{C}}$

$([x_i]\sim \mathfrak{m}_i)$

- Weyl's embedding:

iso. for $n=1$.

$$\iota_{m,n}: S^m\mathbb{A}^n_{\mathbb{C}} \hookrightarrow \mathbb{A}(\mathbb{C}^n\oplus S^2\mathbb{C}^n\oplus\cdots\oplus S^m\mathbb{C}^n)$$
$$\sum_{i=1}^{m}[x_i] \longmapsto (\sigma_1,\dots,\sigma_m)$$

$$\sigma_1 = x_1+\dots+x_m,\quad \sigma_2 = x_1x_2+x_1x_3+\dots+x_{m-1}x_m,\ \dots,\ \sigma_m = x_1\cdots x_m.$$

- Universal characteristic polynomial:

$$\chi_{m,n}: \mathbb{A}^n_{\mathbb{C}}\times S^m\mathbb{A}^n_{\mathbb{C}} \longrightarrow \mathbb{A}(S^m\mathbb{C}^n)$$
$$\Big(x,\ \sum_{i=1}^{m}[x_i]\Big) \longmapsto (x-x_1)\cdots(x-x_m)$$
$$= x^m-\sigma_1x^{m-1}+\dots+(-1)^m\sigma_m$$

- Cayley scheme: $\quad \mathrm{Cayley}_m(\mathbb{A}^n_{\mathbb{C}})=\chi_{m,n}^{-1}(0).$

- Universal spectral cover: $\quad p_{m,n}: \mathrm{Cayley}_m(\mathbb{A}^n_{\mathbb{C}}) \longrightarrow S^m\mathbb{A}^n_{\mathbb{C}}.$

NOT FLAT IN GENERAL FOR $n>1$.

<u>Theorem</u> (Universal Cayley-Hamilton)

$p_{m,n}$ is <u>finite</u>, of <u>degree m</u> and <u>generically étale</u> over $(S^m\mathbb{A}^n_{\mathbb{C}})'\subset S^m\mathbb{A}^n_{\mathbb{C}}$.

For $a=\sum_{i=1}^{s} m_i[x_i]$ $\qquad p_{m,n}^{-1}(a) = \mathrm{Spec}\left(\dfrac{\mathbb{C}[x_1,\dots,x_n]}{\mathfrak{m}_1^{m_1}\cdots\mathfrak{m}_s^{m_s}}\right).$

And, if $a=\mathrm{Spec}(K[\varphi_1,\dots,\varphi_n])$, $\qquad \mathrm{supp}(\mathcal{E})\subset p_{m,n}^{-1}(a)$.

§5. The spectral correspondence

Now come back to the previous situation:

- X smooth proj. var. $/\mathbb{C}$
- $V \to X$ rank r v.b.

$$S^m(V/X) := V \times_{GL_r} S^m \mathbb{A}^r_{\mathbb{C}}$$

$$\begin{tikzcd} S^m(V/X) \arrow[d] \\ X \end{tikzcd} \longleftarrow \text{Fibre bundle with fibres } S^m\mathbb{A}(V_x).$$

$$\mathcal{B}_{m,V} := \{\text{sections of } S^m(V/X)\}.$$

$$S^m\mathbb{A}^r_{\mathbb{C}} \hookrightarrow \mathbb{A}(\mathbb{C} \oplus \dots \oplus S^m\mathbb{C}^r) \rightsquigarrow \iota_{m,V} : \mathcal{B}_{m,V} \hookrightarrow \bigoplus_{i=1}^m H^0(X, S^iV).$$

We get a <u>factorization</u>:

$$\begin{tikzcd} \mathcal{M}_{m,V} \arrow[r, "sd"] \arrow[rr, bend left, "h_{m,V}"] & \mathcal{B}_{m,V} \arrow[r, hook, "\iota_{m,V}"] & \bigoplus_{i=1}^m H^0(X, S^iV) \\ (E,\varphi) \arrow[r, mapsto] & {[x \mapsto sd(\varphi_1(x), \dots, \varphi_r(x))]} & \end{tikzcd}$$

$\to$ <u>Study the fibres of sd.</u>

Idea: Take $b \in \mathcal{B}_{n,V}$.

$$\begin{tikzcd}
V \arrow[drr] & Y_b \arrow[l, hook'] \arrow[r] \arrow[dr, "\pi"'] & \mathrm{Cayley}_n(V/X) := V \times_{GL_r} \mathrm{Cayley}_n(\mathbb{A}^r_{\mathbb{C}}) \arrow[d] \\
 & X \arrow[r, "b"] & S^n(V/X) := V \times_{GL_r} S^n \mathbb{A}^r_{\mathbb{C}}
\end{tikzcd}$$

$Y_b \xrightarrow{\pi} X$ ← spectral cover

- If b is generically mult. free, π is generically étale.

Problem: π is not flat in general.

Solutions:

- If $\mathrm{rk}\, V = 1$, then it is flat.
- For $\dim X = 2$, $V = \Omega^1_X$ ⇝ Chan-Ngo, Cohen-Macaulay modification.
- For $\dim X = 1$, $\mathrm{rk}\, V > 1$:

$$\pi_* \mathcal{O}_{Y_b} = \underset{\substack{\uparrow \\ \text{locally free}}}{F} \oplus \underset{\substack{\uparrow \\ \text{torsion}}}{T} \rightsquigarrow \tilde{Y}_b = \underline{\mathrm{Spec}}(F) \subset Y_b$$

$$\tilde{\pi} = \pi|_{\tilde{Y}_b}.$$

$\tilde{\pi} : \tilde{Y}_b \longrightarrow X$ is finite locally free.

Moreover, Y_b irreducible $\Rightarrow$ $\tilde{Y}_b$ integral.

Thm. If b is generically multiplicity free and Y_b is irreducible:

$$h^{-1}(b) = \left\{ \begin{array}{l} \text{Torsion free sheaves of} \\ \text{generic rk. 1 over } \tilde{Y}_b \end{array} \right\} \Big/ \text{iso.}$$

Moreover, if $\tilde{Y}_b$ is smooth:

$$h^{-1}(b) = \mathrm{Pic}(\tilde{Y}_b)$$

components are abelian varieties of dim. $g_{\tilde{Y}_b}$.

Idea of the proof.

(E,φ) V-twisted Higgs bundle on X $\longleftrightarrow$ $\mathcal{E}$ coherent sheaf on V.

$sd(E,\varphi) = b$ $\longleftrightarrow$ $\mathrm{supp}(\mathcal{E}) = Y_b \subset V$.

Universal Cayley-Hamilton

§6. Properties of the spectral curve

Problem: For which $b \in \mathcal{B}_{n,V}$ is $\tilde{Y}_b$ integral and smooth?

- BNR $\to$ For $\mathrm{rk}\, V = 1$ apply Bertini since

  b is a section of some line bundle.

- We study the problem for $\mathrm{rk}\, V = 2$ and (E, φ) of type

  SL_2, that is, $\det E = \mathcal{O}_X$, $\mathrm{tr}\, \varphi = 0$.

  In this case,

$$\mathcal{B} = \{ b \in H^0(X, S^2 V) \mid \text{locally } b = (b_1, b_2, b_3),\ b_3^2 = b_1 b_2 \}.$$

↑ sections of some "bundle of cones".

Study the vanishing locus of $b \in \mathcal{B}$.

Thank you! :)

Questions?