A finiteness theorem concerning compact analytic varieties

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Translator's note

The translator (*) takes full responsibility for any errors introduced in the passage from one language to another, and claims no rights to any of the mathematical content herein.

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MATHEMATICAL ANALYSIS. – *A finiteness thoerem concerning compact analytic varieties.* Note by HENRI CARTAN and JEAN-PIERRE SERRE, presented by Jacques Hadamard.

THEOREM. – *Let* X *be a compact complex analytic variety. Let* $\mathscr{F}$ *be a coherent analytic sheaf* ([1]) *over* X. *Then the cohomology groups* $H^q(X, \mathscr{F})$ *(*q *an integer* ≥ 0*) are complex vector spaces of finite dimension.*

This result is true in the particular case where $\mathscr{F}$ is the sheaf of germs of holomorphic sections of an analytic bundle E over X, where the fibre is a complex vector space of finite dimension ([2]). A sheaf like that is locally isomorphic to the sheaf $\mathscr{O}^r$ of r-tuples of germs of holomorphic functions (r denoting the dimension of the fibre of E).

1. Before proving the theorem above, we give some preliminary definitions. An open subset V of X is called *adapted* to $\mathscr{F}$ if V is an Stein manifold ([1]) and if there exists a finite system of p sections $s_i \in H^0(V, \mathscr{F})$ that generate $\mathscr{F}_x$ in every point $x \in V$. Every sufficiently small Stein open subset is adapted to $\mathscr{F}$. If V is adapted to $\mathscr{F}$, $\mathscr{F}$ is identified, on V, with the quotient of the sheaf $\mathscr{O}^p$ by a subsheaf $\mathscr{R}$, which is

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[1] Cf. *Séminaire Ec. Norm. Sup.*, 1951-1952, "exposés" XVIII and XIX, aswell as the conference by H. CARTAN, *Colloque de Bruxelles sur les fonctions de plusieurs variables* (March 1953).

[2] In this particular case, the theorem has already been proven by K. Kodaira (under slightly more restrictive hypotheses), thanks to a generalization of the theory of harmonic forms. *Cf.* K. KODAIRA, *Proc. Nat. Acad. Sc. U.S.A.*, **39**, 1953 (to appear).

coherent, since $\mathscr{F}$ is coherent. Thus $H^q(V, \mathscr{R}) = 0$ for $q > 0$ ([1]). As a consequence, the sequence

$$0 \longrightarrow H^0(V, \mathscr{R}) \longrightarrow H^0(V, \mathscr{O}^p) \longrightarrow H^0(V, \mathscr{F}) \longrightarrow 0$$

is exact. We endow $H^0(V, \mathscr{O}^p)$ with the topology of compact convergence; it is a Fréchet space (i.e. locally convex, metrizable and complete). $H^0(V, \mathscr{R})$ is closed ([3]) in $H^0(V, \mathscr{O}^p)$, so the quotient space $H^0(V, \mathscr{O}^p)/H^0(V, \mathscr{R})$ is a Fréchet space. This defines a topology over $H^0(V, \mathscr{F})$, and we easily see that it does not depend on the choice of the s_i.

Well understood, if $\mathscr{F}$ is isomorphic to $\mathscr{O}^p$ over V, the topology of $H^0(V, \mathscr{F})$ is the compact convergence topology.

LEMMA. – *Let $\mathscr{F}$ be a coherent analytic sheaf over a complex analytic variety X; let V and V' two open subsets of X adapted to $\mathscr{F}$, such that $V \subset V'$. Then the map $\varphi : H^0(V', \mathscr{F}) \to H^0(V, \mathscr{F})$ is continuous. Moreover, if the closure of V is compact and contained in V', then φ is completely continuous.*

The first point is obvious. The second results from the fact that every set of holomorphic functions on V' bounded over $\bar{V}$ induce in V a relatively compact set.

2. Let $\mathfrak{U} = (U_i)_{i \in I}$ a finite covering of the compact variety X by open subsets U_i adapted to $\mathscr{F}$. For every integer $q \geq 0$, we associate to every tuple $(i_0, \dots, i_q)$ of indices of I a section $f_{i_0, \dots, i_q}$ of $\mathscr{F}$ over $U_{i_0 \dots i_q} = U_{i_0} \cap \dots \cap U_{i_q}$, with an alterating dependence on the indices. This tuples $(f_{i_0 \dots i_q})$ form a vector space $C^q(\mathfrak{U}, \mathscr{F})$. The topology of $H^0(U_{i_0 \dots i_q}, \mathscr{F})$ obtained by the procedure of nº 1, define over $C^q(\mathfrak{U}, \mathscr{F})$ a Fréchet space topology. We define in the usual way a coboundary operator $\delta : C^q(\mathfrak{U}, \mathscr{F}) \to C^{q+1}(\mathfrak{U}, \mathscr{F})$, which is continuous after the lemma. The kernel $Z^q(\mathfrak{U}, \mathscr{F})$ of δ is a Fréchet space. We denote by $H^q(\mathfrak{U}, \mathscr{F})$ the cohomology spaces of the complex $\{C^q(\mathfrak{U}, \mathscr{F}), \delta\}$.

3. Take now two open coverings $\mathfrak{U} = (U_i)$ and $\mathfrak{U}' = (U'_i)$ so that $\bar{U}_i \subset U'$ and the U_i and U'_i are open subsets adapted to $\mathscr{F}$. The linear maps

$$H^q(\mathfrak{U}', \mathscr{F}) \xrightarrow{\rho} H^q(\mathfrak{U}, \mathscr{F}) \longrightarrow H^q(X, \mathscr{F})$$

are (algebraic) isomorphisms, since ([4]) the cohomology groups $H^p(U'_{i_0, \dots, i_q}, \mathscr{F})$ and $H^q(U_{i_0, \dots, i_q}, \mathscr{F})$ are trivial for $p > 0$. Now it suffices to prove that $H^q(\mathfrak{U}, \mathscr{F})$ has finite dimension.

The lemma implies that the map $r : Z^q(\mathfrak{U}', \mathscr{F}) \to Z^q(\mathfrak{U}, \mathscr{F})$ is completely continuous. Let then E be the product space $C^{q-1}(\mathfrak{U}, \mathscr{F}) \times Z^q(\mathfrak{U}', \mathscr{F})$, F the space $Z^q(\mathfrak{U}, \mathscr{F})$, u the map (δ, r) from E to F, and v the map $(0, -r)$. Since ρ is an isomorphism, u maps E onto F; a theorem of L. Schwartz ([5]) now shows that the image of $u + v = (\delta, 0)$ is a closed subspace of finite dimension of F. This shows that $H^q(\mathfrak{U}, \mathscr{F})$, and thus also $H^q(X, \mathscr{F})$, has finite dimension.

[3] *Cf.* H. CARTAN, *Ann. Ec. Norm. Sup.*, **61**, 1944, p. 149-197 (first corollary to theorem α, p. 194).

[4] This result is not explicit in the bibliography; it is proven by a analogous method to that used by A. Weil in his proof of the de Rham theorems. (*Comm. Math. Helv.*, **26**, 1952, p. 119-145).

[5] *Comptes rendus*, **236**, 1953, p. 2472 (corollary to theorem 2).