Multiplicative Hitchin Fibrations and Langlands duality

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§1. MULTIPLICATIVE HITCHIN FIBRATION

C complex projective curve

G semisimple complex group

$G^{sc} \longrightarrow G$ simply connected cover

Definition A *multiplicative G-Higgs bundle* over C is a pair (E,φ):

- $E \to C$ principal G-bundle
- $\varphi \in \Gamma\left(C \setminus \{p_1, \dots, p_n\}, E \times^{Ad(G)} G^{sc}\right)$

Local type: z local coord. near p_i,

$$\varphi|_{\mathbb{D}_{p_i}} \in G[[z]] \backslash G^{sc}((z)) / G[[z]] \cong X_*(T^{sc})/W \cong X_+(T^{sc})$$

[Fix:
$$\begin{tikzcd} T^{sc} \arrow[r, hook] \arrow[d] & B^{sc} \arrow[r, hook] & G^{sc} \arrow[d] \\ T \arrow[r, hook] & B \arrow[r, hook] & G \end{tikzcd}$$
]

STACK OF MULTIPLICATIVE G-HIGGS BUNDLES

$$\mathcal{M}_{G,D} = \left\langle (E,\varphi) \text{ mult. } G\text{-Higgs bundle on } C \text{ w. local type } D \right\rangle$$

$$D = \sum_{i=1}^{n} \lambda_i p_i, \qquad \lambda_i \in X_+(T^{sc}).$$

$(r = \mathrm{rk}\, G)$

- $\omega_1, \dots, \omega_r$ fundamental weights of G^{sc}

$\Updownarrow$

$\rho_1, \dots, \rho_r$ fundamental representations of G^{sc}

$g \in G^{sc}$

$$p_i(g) := tr\left(\rho_{\omega_i}(g)\right)$$

MULTIPLICATIVE HITCHIN FIBRATION (Hurtubise – Markman)

$$h_{G,D} : \mathcal{M}_{G,D} \longrightarrow \mathcal{A}_{G,D} := \bigoplus_{i=1}^{r} H^0\left(C, \mathcal{O}_C\left(\langle D, \omega_i \rangle\right)\right), \qquad \langle D, \omega_i \rangle = \sum_j \langle \lambda_j, \omega_i \rangle p_j$$

$$(E,\varphi) \longmapsto \left(p_1(\varphi), \dots, p_r(\varphi)\right)$$

Remarks

1. Global version of $[G^{sc}/G] \longrightarrow G^{sc}/\!\!/G \cong T^{sc}/W \overset{\text{simply connected!!}}{\cong} \mathbb{A}^r$

2. Contrast with usual Hitchin fibration: $[\mathfrak{g}/G] \longrightarrow [\mathfrak{g}/\!\!/G]$.

3. There is a Hitchin–Kobayashi correspondence:
(Charbonneau – Hurtubise, Smith)

$$\mathcal{M}^{ps}_{G,D} \cong \{\text{singular } K\text{-monopoles on } C\times S^1\}/_{\text{gauge}}$$

$K \leq G$ maximal compact

[solutions to $F_\nabla = *\nabla\Phi$ Bogomolny eqn.]

local types $\longleftrightarrow$ Dirac-type singularities

4. When $C = \mathbb{C}, \mathbb{C}^*$ or elliptic curve we have

($\mathbb{CP}^1$ w. framings)

$$\boxed{C\times S^1 \cong \mathbb{R}^3/\mathbb{Z}^{1,2,3}}$$

$\mathcal{M}_{G,D}$ hyperkähler and Mochizuki's NAHT:

$$\mathcal{M}_{G,D} \cong \begin{cases} \text{difference modules} \\ q\text{-difference modules} \\ \text{elliptic difference modules} \end{cases}$$

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§2. THE MONOID POV

Idea: Want: $G^D \longrightarrow C$, partial compactification of G^{sc} over C,

$$\mathcal{M}_{G,D} = \mathrm{Map}_c(C,[G^D/G]) \qquad \mathcal{A}_{G,D} = \mathrm{Map}_c(C, G^D/\!\!/G).$$

[similar to $\mathrm{Higgs}_G = \mathrm{Map}_c(C,[\mathfrak{g}^{K_C}/G])$, $\mathrm{Hitch}_G = \mathrm{Map}_c(C, \mathfrak{g}^{K_C}/\!\!/G)$].

Solution : The Vinberg monoid (Frenkel-Ngô, Bouthier, Chi, Wang)

$$\begin{tikzcd}
\dfrac{G^{sc}\times T^{sc}}{Z_{G^{sc}}} \arrow[r, hook] \arrow[d, "\alpha"'] & G\times G \circlearrowright \mathrm{Vin}(G^{sc}) \circlearrowleft T \arrow[d] \\
T \circlearrowright T^{ad} \arrow[r, hook] & \mathbb{A}^r \circlearrowleft T
\end{tikzcd}$$

$\nwarrow$ flat family of copies of G^{sc} degenerating at hyperplanes $\{\alpha_i = 0\}$

$\left[\lim_{z\to 0} z^{\lambda} \in \mathbb{A}^r \Leftrightarrow \lambda \in X_+(T^{sc})\right]$

$$\begin{tikzcd}
G^D \arrow[r] \arrow[d] & \left[\mathrm{Vin}(G^{sc})/T\right] \arrow[d] \\
C \arrow[r, "{(\mathcal{O}_C(D),\, 1)}"'] & \left[\mathbb{A}^r/T\right]
\end{tikzcd}$$

Example : $\mathrm{Vin}(SL_2) \cong \mathrm{Mat}_{2\times 2} \xrightarrow{\det} \mathbb{A}^1$

$$\begin{tikzcd}
SL_2^D \arrow[r] \arrow[d] & \left[\mathrm{Mat}_{2\times 2}/\mathbb{G}_m\right] \arrow[d, "\det"] \\
C \arrow[r, "{(\mathcal{O}_C(D),\, 1)}"'] & \left[\mathbb{A}^1/\mathbb{G}_m\right]
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathcal{M}_{SL_2,D} \arrow[d, "tr"] \\
\mathcal{A}_{SL_2,D}
\end{tikzcd}$$

$$\mathcal{M}_{SL_2,D} = \left\langle (E,\varphi) \;\middle|\; \begin{array}{ll} E\to C \text{ rk. 2 v.b.} & \det E \cong \mathcal{O}_C \\ \varphi : E \to E\otimes \mathcal{O}_C(D) & \det \varphi = 1 \end{array} \right\rangle$$

$$\mathcal{A}_{SL_2,D} = H^0(C, \mathcal{O}_C(D))$$

§ 3. SYMMETRIES AND CAMERAL DESCRIPTION

$$I_{G,D,x} = \{ g \in G \mid gxg^{-1} = x \}$$

$$\begin{tikzcd} I_{G,D} \arrow[r] \arrow[d, squiggly] & G^D \arrow[d] \\ \boxed{J_{G,D}} \arrow[r] & G^D /\!/ G \end{tikzcd} \qquad \begin{tikzcd} J_{G,D,a} \arrow[r] \arrow[d] & J_{G,D} \arrow[d] \\ C \arrow[r, "a"'] & G^D /\!/ G \end{tikzcd}$$

REGULAR CENTRALIZER

$a \in \mathcal{A}_{G,D}$

PICARD STACK ACTING ON FIBRES

$$\boxed{\mathcal{P}_{G,D,a} = \mathrm{Bun}_{J_{G,D,a}/C}} \circlearrowright h^{-1}_{G,D}(a)$$

↳ Has a "Galois" description in terms of the

CAMERAL COVER

$$\begin{tikzcd} \tilde{C}_a \arrow[r] \arrow[d] & T^D \arrow[d] \\ C \arrow[r, "a"] & T^D/W \cong G^D /\!/ G \end{tikzcd}$$

f.g. abelian group

abelian variety

$$\mathcal{P}_{G,D,a} \cong \Lambda \times P \times BH$$

gp. of mult. type

Thm [Hurtubise–Markman, Bouthier, Chi, Wang]

Assume:

- D "ample"
- a $\underbrace{\text{"generic"}}$

group-like locus / HM-regular

↳ cut boundary or discriminant but not both

Then:

- $\tilde{C}_a$ is smooth
- $\mathcal{P}_{G,D,a}$ is a Beilinson 1-Motive
- $h^{-1}(a)$ is a $\mathcal{P}_{G,D,a}$-torsor (trivializable)
- $\tilde{C}_a \to C$ cameral cover of type $\mathfrak{g}$ and $J_{G,D,a} \cong J_G$

§ 4. REVIEW OF DONAGI-PANTEV DUALITY

$\mathfrak{g}$ complex semisimple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ Cartan, W Weyl group

$\tilde{C} \xrightarrow{\pi} C$ ramified Galois W-cover of smooth curves

Def: This is a <u>cameral curve of type $\mathfrak{g}$</u> if it is étale-locally iso. to p.b. of

$$\mathfrak{t} \longrightarrow \mathfrak{t}/W \quad \leftarrow \text{standard cameral cover}$$

has simple Galois ramification if $\mathrm{im}(C \to \mathfrak{t}/W)$ intersects transversely the discriminant divisor $\mathcal{D} \subset \mathfrak{t}/W$.

G complex semisimple group w. $\mathrm{Lie}(G) = \mathfrak{g}$ | $G^\vee$ Langlands dual

$\underbrace{J_G \to C}$ group scheme defined as $J_G \underset{\text{open}}{\hookrightarrow} \pi_*(\tilde{C} \times T)^W$

closed

(Donagi-Gaitsgory)

[on ramification points x

$\mathrm{Ker}\, \alpha_x = J_{G,x} \hookrightarrow T^{s_{\alpha_x}}$]

The regular centralizers of usual Hitchin fibrations are like this. $\longrightarrow$

$$\begin{tikzcd} \mathcal{P}_G = \mathrm{Bun}_{J_G/C} \arrow[r, phantom, "\circlearrowright"] \arrow[dr] & \mathrm{Higgs}_G \arrow[d] \\ & \mathrm{Hitch}_G \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{t} \arrow[rr, "\text{Killing form}", "\sim"'] \arrow[dr] & & \mathfrak{t}^* \arrow[dl] \\ & \mathfrak{t}/W \cong \mathfrak{t}^*/W & \end{tikzcd}$$

$\rightsquigarrow$ Hitchin bases and cameral covers for dual groups are identified

Thm [Donagi–Pantev]

$a \in \mathrm{Hitch}^{\diamond}_G$ ← smooth cameral curve
$\cong$
$\mathrm{Hitch}^{\diamond}_{G^\vee}$

$$\boxed{\mathcal{P}_{G,a} = \mathrm{Dual}\left(\mathcal{P}_{G^\vee,a}\right)}$$

→ Topological argument:
$$H_1(P_{G,a}) \cong H_1(P_{G^\vee,a})^\vee$$

[For char $\neq 0$, using Tate modules by Chen and Zhu]

$$\mathrm{Coh}\left(\mathrm{Higgs}_G / \mathrm{Hitch}^{\diamond}_G\right) \xrightarrow[\sim]{FM} \mathrm{Coh}\left(\mathrm{Higgs}_{G^\vee} / \mathrm{Hitch}^{\diamond}_G\right)$$

⊛ compatible w. abelianized Hecke–Wilson ('t Hooft)

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§5 MULTIPLICATIVE DUALITY. SIMPLY-LACED CASE

In the simply laced case, duality follows from Donagi–Pantev.

$$\begin{tikzcd} T^{sc} \arrow[r, "\sim"] \arrow[d] & (T^\vee)^{sc} \arrow[d] \\ T^{sc}/W \arrow[r, "\sim"] & (T^\vee)^{sc}/W \end{tikzcd} \longrightarrow$$

Thm [G.] If G is simply laced

$$\boxed{\mathcal{P}_{G,a} = \mathrm{Dual}\left(\mathcal{P}_{G^\vee,a}\right)}$$

$\mathcal{M}_{G,D,a}$ $\qquad$ $\mathcal{M}_{G^\vee,D,a}$

Example

$$\mathcal{M}_{SL_2,D} = \left\langle (E,\varphi) \;\middle|\; \begin{array}{l} E \to C \text{ rk. } 2,\ \det E \xrightarrow{\sim} \mathcal{O}_C \\ \varphi : E \to E \otimes \mathcal{O}(D),\ \det\varphi = 1 \end{array} \right\rangle$$

$$\downarrow\ tr = h_D$$

$$\mathcal{A} = H^0(C, \mathcal{O}(2D))$$
$\ni a$

SPECTRAL CURVE → $S_a = \{ y^2 + ay + 1 = 0 \}$
$\downarrow$
C

$$\mathcal{P}_{SL_2,a} \cong \mathrm{Prym}(S_a/C) \circlearrowright h_D^{-1}(a)$$
$\boxed{\text{BNR}}$

$$\mathcal{M}_{PGL_2,D} = \left[\mathcal{M}_{SL_2,D} / Jac(C)[2]\right] \rightsquigarrow \mathcal{P}_{PGL_2,a} = Prym(S_a) / Jac(C)[2] = Prym(S_a)^\vee$$

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§ 5 MULTIPLICATIVE DUALITY. NON SIMPLY-LACED CASE

[Excluding $A^{(2)}_{2\ell}$]

In this case T and $T^\vee$ are not matched W equivariantly

But: If $H = G^\theta$ with

[G simply laced simply connected
θ diagram automorphism]

↳ classified by affine Dynkin diagrams

Then,

$$\boxed{G\theta /\!/ G \cong (H^\vee)^{sc} /\!/ H^\vee} \cong \frac{T/(1-\theta)(T)}{W^\theta}$$

[Mohrdieck]

<u>Def.</u> Let (G,θ) as above.

A <u>θ-twisted multiplicative G-Higgs bundle</u> on C is (E,φ)

- $E \to C$ principal G-bundle
- $\varphi \in \Gamma\left(C \setminus \{p_1,\dots,p_m\},\ E \times_{\boxed{G,\theta}} G\right)$

twisted adjoint action
$g * x = g x \theta(g)^{-1}$

[same local types as untwisted G-Higgs]

[Griffin Wang, unpublished] $\rightsquigarrow$ Study of regular centralizers for $G \underset{\theta}{\curvearrowright} G$

↳ cameral covers of type h

↳ $J_{(G,\theta)} \cong J_H$ $\left[\cong \pi_*\left(\tilde{C}_a \times T_H\right)^{W^\theta}\right]$

Thm [G.] For (G,θ) as above, $H = G^{\theta}$, [G^{θ} is simply connected]

$$\boxed{\mathcal{P}_{(G,\theta),a} = \mathrm{Dual}\left(\mathcal{P}_{H^{\vee},a}\right)}$$

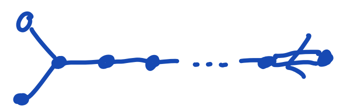
$\mathcal{M}_{(G,\theta),D,a}$ $\qquad$ $\mathcal{M}_{H^{\vee},D,a}$

$(H^{\vee})^{sc} = G_{\theta}$

$T \longrightarrow T_{\theta}$

Examples

$(SL_{2\ell}, SL_{2\ell}\theta) \leftrightarrow (SO_{2\ell+1}, Spin_{2\ell+1})$



$A_{2\ell-1}^{(2)} = (B_{\ell}^{(1)})^{\vee}$

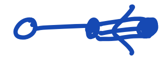
$(Spin_{2\ell+2}, Spin_{2\ell+2}\theta) \leftrightarrow (PSp_{2\ell}, Sp_{2\ell})$

$D_{\ell+1}^{(2)} = (C_{\ell}^{(1)})^{\vee}$

$(E_6, E_6\theta) \leftrightarrow (F_4, F_4)$

$E_6^{(2)} = (F_4^{(1)})^{\vee}$

$(Spin_8, Spin_8\theta_3) \leftrightarrow (G_2, G_2)$



$D_4^{(3)} = (G_2^{(1)})^{\vee}$

$\boxed{\text{Special cases}}$

$(SL_3, SL_3\theta)$

$A_2^{(2)}$ ⊛

$(SL_{2\ell+1}, SL_{2\ell+1}\theta)$

$A_{2\ell}^{(2)}$

[Physics: Discrete theta-angle]

⊛ Explicit description in the paper: **Bilinear pairs**: (E,φ)

$\varphi : E \longrightarrow E^{*} \otimes \mathcal{O}(D).$

$\downarrow$

Spectral description and "Prym variety".

THANK YOU