BRUHAT-TITS THEORY. SEMINAR SCRIPT

1. Some motivation

1.1. **Parahoric torsors.** There are many reasons to study Bruhat-Tits theory, but from the point of view of our research group, a main motivation would be to understand the theory of parahoric torsors on curves. When studying the moduli theory of bundles on a curve C defined over an algebraically closed field k, one can be naturally led to consider the space of $\mathcal{G}$-bundles on C, where $\mathcal{G}$ is a group scheme over C which might not be constant in general. For example, these non-constant group schemes arise naturally in the following problems

(1) moduli of vector bundles with fixed non-trivial determinant,
(2) moduli of parabolic bundles,
(3) moduli of Hecke transformations,
(4) Prym varieties associated to ramified covers of curves,
(5) moduli of equivariant bundles.

The complete local field of the curve C at a point p is isomorphic to the ring of formal power series $\mathcal{O} = k[[t]]$, whose fraction field is the field of formal Laurent series $K = k((t))$. There is a naturally defined valuation on K, sending a formal Laurent series to its highest pole order. The ring of integers then is $\mathcal{O}$ and the residue field is k. The non-constant group schemes $\mathcal{G}$ from the above examples are locally determined by group schemes over $\operatorname{Spec} \mathcal{O}$. Restricting to $\operatorname{Spec} K$ we obtain a group scheme $\mathbf{G}$ over K.

1.2. **Symmetric varieties and buildings.** For those more inclined towards differential geometry, it can be also interesting to explore the parallels between the theory of real symmetric spaces and the theory of Bruhat-Tits buildings associated to p-adic groups.

2. Bruhat-Tits theory

Bruhat-Tits theory is the theory of reductive groups over local fields. In this seminar, by a local field we will understand a field K, endowed with a nontrivial discrete valuation $v : K \longrightarrow \Gamma \subset \mathbb{R}$. We will denote by $\mathcal{O} \subset K$ the ring of integers, that is $\mathcal{O} = \{a \in K : v(a) \geq 0\}$, by $\mathfrak{m} = \{a \in \mathcal{O} : v(a) > 0\}$ the corresponding prime ideal, and by $k = \mathcal{O}/\mathfrak{m}$ the residue field. We will always assume that K is complete (with respect to the topology induced by the valuation), and that k is a perfect field (for example, we can assume that k is finite, that it is algebraically closed, or that it is of characteristic 0). We will study the structure of linear algebraic groups $\mathbf{G}$ over K such that their neutral connected component $\mathbf{G}^\circ$ is a reductive group.

Such a group $\mathbf{G}$ over K can be *split*, in which case it will be just a base change $\mathbf{G} = G_K$, for G a reductive group over k, and we will have $\mathbf{G}(K) = G(K)$. More generally, $\mathbf{G}$ can be *quasisplit*, which means that it splits under a finite ramified extension $K'|K$. I suggest to study and understand first the split case and afterwards try to consider the quasisplit case.

To the group $\mathbf{G}$ one can associate a combinatorial structure $\mathcal{B}(\mathbf{G})$ called the (Bruhat-Tits) *building*. One of the main results of Bruhat-Tits theory is that there is a way of uniquely associate group schemes $\mathcal{G}_\Omega$ over $\mathcal{O}$ to compact subset Ω of the building, in such a way that $\mathcal{G}_\Omega|_K = \mathbf{G}$, and $\mathcal{G}_\Omega(\mathcal{O}) \subset \mathbf{G}(K)$ is a certain subgroup of $\mathbf{G}(K)$ determined by the data Ω. For example, if $\mathbf{G} = G_K$ is split, to a dominant cocharacter λ of a maximal torus $T \subset G$ one can associate a group scheme $\mathcal{G}_\lambda$ such that $\mathcal{G}_\lambda|_0$ is isomorphic to the standard parabolic subgroup $P_\lambda \subset G$ associated to λ. The subgroup $\mathcal{G}_\lambda(\mathcal{O}) \subset \mathbf{G}(K)$ is called the *standard parahoric subgroup* associated to λ.

3. Schedule

- **Session 1. Review of reductive groups**. It is appropriate to start by reviewing all the basics about linear algebraic groups, and in particular, about reductive groups. We should review the Levi decomposition, the notions of parabolic and Borel subgroup, root data, pinnings, etc. Focusing on examples is crucial for understanding. I recommend doing GL_n (in general) and maybe something like Sp_4.
- **Session 2. Affine apartment and affine root system**. The idea is to cover the entire Section 1 of [Corvallis] but skipping all the parts where the group is not assumed to be split. Essentially this is 1.1, 1.5-1.9, 1.13 and the example 1.14.
- **Session 3. The building**. Cover Section 2 of [Corvallis]. Again, everything making reference to non-split groups (for example, Section 2.6 and Example 2.10) can be skipped. Section 2.3 about the metrics is interesting but can be postponed. In parallel, the speaker can take ideas and examples from Section 2 of [Casselman] and Section 4 of [Rabinoff].
- **Session 4. Structure theory**. Talk about BN-pairs, Bruhat/Cartan/Iwasawa decompositions and parahorics. In principle this is 3.1-3.3 in [Corvallis]. It would be nice to see some of the proofs or arguments behind the decompositions. At the very least we should do Section 1 of [Casselman]. Section 3.3 of [Rabinoff] could also be OK. Maybe we should divide this one in two, and make parahorics a different talk.
- **Session 5. Bruhat-Tits group schemes**. Sections 3.4-5 and Example 3.10 in [Corvallis]. Maybe it would be interesting to gather more examples from elsewhere. Probably it does not make sense to look at the proofs in this talk, but could make sense in a future talk.
- **Session 6. Parahorics à la Balaji-Seshadri**. Prove Theorem 2.3.1 in [Balaji-Seshadri], that shows that parahoric subgroups can be obtained as local automorphism groups of equivariant bundles.
- **Sessions 7 and beyond.** TBD.

4. Reference list

Corvallis *Reductive groups over local fields.* J. Tits. In "Automorphic Forms, Representations and L-functions" (usually referred to as the Corvallis volume). — This is the typically recommended reference; and the one we will try to follow. It is maybe too focused in the quasisplit examples which, although they are also interesting from the geometric perspective, we can omit in a first approach to the topic.

Bruhat-Tits *Groupes réductifs sur un corps local.* Volumes I and II. F. Bruhat and J. Tits. — This is the classical and the foundational reference, many proofs are still only found here. Maybe not so good for just reading, but rather for consulting.

Rabinoff *The Bruhat-Tits building of a p-adic Chevalley group and an application to representation theory.* Honors thesis of J. Rabinoff. https://services.math.duke.edu/~jdr/papers/building.pdf — A very nice and expository reference. Good for a first approach to the topic. An important issue is that it takes the approach of starting with abstract root data (obtained from a complex semisimple group) and then constructing a Chevaley group over any field and obtaining the non-split forms as "generalized Levi subgroups", while typically one starts with a reductive group (which might be non-split) over the local field K.

Casselman *The Bruhat-Tits tree of $SL(2)$.* Notes by B. Casselman. https://ncatlab.org/nlab/files/CasselmanOnBruhatTitsTree2014.pdf. — Very nice notes, specially centered in SL_2, which makes them ideal to get a hands-on introduction to the topic.

Balaji-Seshadri *Moduli of parahoric $\mathcal{G}$-torsors on a compact Riemann surface.* Paper by V. Balaji and C.S. Seshadri. https://arxiv.org/abs/1009.3485. — In this paper, Balaji and Seshadri study $\mathcal{G}$-torsors over a compact Riemann surface, where $\mathcal{G}$ is a Bruhat-Tits group scheme locally determining parahoric subgroups of $G(K)$. In particular, they show how one is naturally led to consider these objects when studying equivariant bundles.

Heinloth *Introduction to parahoric torsors.* Talks by J. Heinloth in Edinburgh. https://media.ed.ac.uk/media/Jochen+Heinloth+Part+1/1_qrtisp9f. — In this talk, Heinloth explains some of the basics of Bruhat-Tits theory, and gives some examples coming from moduli theory where one needs to apply it.