Universal Spectral Covers & the Hitchin System

By: Guillermo Gallego
(UCM - ICMAT)

Commuting Families of Endomorphisms

$\Bbbk$ — alg. closed field

E $\Bbbk$-vector space of dim. n

$\varphi_1, \dots, \varphi_r \in \operatorname{End} E$, $[\varphi_i, \varphi_j] = 0$

$$A = \Bbbk[\varphi_1, \dots, \varphi_r]$$

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evaluation morphism

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$\Bbbk[X_1, \dots, X_r]$-module structure on E

Commuting Families of Endomorphisms

k — alg. closed field

E k-vector space of dim. n

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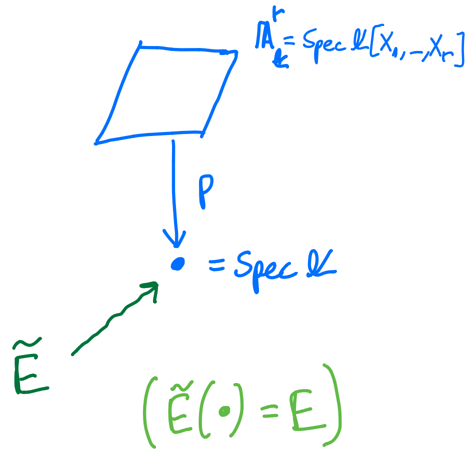
$$k[X_1, \dots, X_r] \longrightarrow A \subset \operatorname{End} E$$
$$p(X_1, \dots, X_r) \longmapsto p(\varphi_1, \dots, \varphi_r)$$

evaluation morphism

$k[X_1, \dots, X_r]$-module structure on E

Geometric picture:



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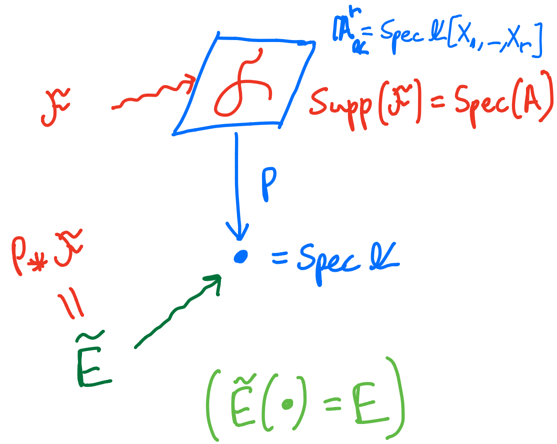

Commuting Families of Endomorphisms

(Case $r=1$)

$k[T]$ is a PID

$$A = k[\varphi] = k[T] \Big/ \operatorname{Ker}\Big(\underset{p(T) \mapsto p(\varphi)}{k[T] \to A} \Big) = k[T] \big/ (m_\varphi)$$

$m_\varphi :=$ minimal polynomial of φ

Roots of $m_\varphi \equiv$ Eigenvalues of φ: $x_1, \ldots, x_s \in \mathbb{A}^1_k$

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Spectral decomposition: $E = \bigoplus_{i=1}^{s} E_i$, $E_i = \operatorname{Ker}(\varphi - x_i \, id_E)^{\alpha_i}$

Characteristic polynomial: $p_\varphi(T) = (T - x_1)^{m_1} \dots (T - x_n)^{m_s}$

$m_i = \dim(E_i)$

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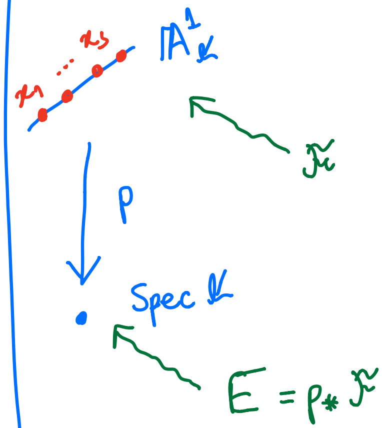

$\operatorname{Supp} \mathcal{F} = \operatorname{Spec}(A) \subset \operatorname{Spec}\left(\Bbbk[T] / (p_\varphi)\right)$

Commuting Families of Endomorphisms

(Case $r > 1$)

$$A = \mathbb{K}[\varphi_1, \ldots, \varphi_r] = \mathbb{K}[X_1, \ldots, X_r] / I$$

$$I = \operatorname{Ker}\begin{pmatrix} \mathbb{K}[X_1, \ldots, X_r] \to A \\ p(X_1, \ldots, X_r) \mapsto p(\varphi_1, \ldots, \varphi_r) \end{pmatrix}$$

← "minimal polynomial"

Generated by several equations !!

Primary decomposition

$$I = \mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_s^{\alpha_s}$$

$$\Downarrow$$

$$E = \bigoplus_{i=1}^{s} E_i \quad , \quad E_i = \operatorname{Ker}\left(\mathfrak{m}_i^{\alpha_i}\right)$$

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Cayley-Hamilton generalized: $\alpha_i \leq n_i$ (Nakayama)

$$\Downarrow$$

$J = m_1^{n_1} \cdots m_s^{n_s}$ "characteristic polynomial" $\quad J \subseteq I$

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Generated by several equations !!

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$I = \mathfrak{m}_1^{\alpha_1}\cdots\mathfrak{m}_s^{\alpha_s}$

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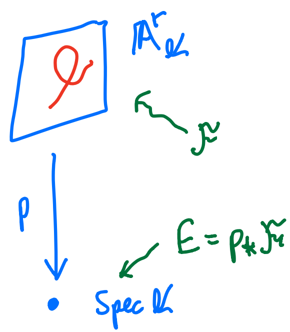

$\mathrm{Supp}\,\mathcal{F} = \mathrm{Spec}(A) \subseteq \mathrm{Spec}\left(\Bbbk[X_1,\dots,X_r]/J\right)$

The Spectral Data map

$$\mathrm{sd}: \left\{\begin{array}{c}\text{Commuting}\\ \text{families of } r\\ \text{end. of } E\end{array}\right\} \longrightarrow \mathrm{Chow}_n(\mathbb{A}^r_{\mathbb{k}}) = (\mathbb{A}^r_{\mathbb{k}})^n / \mathfrak{S}_n$$

$$A \longmapsto [x_1^{m_1}, \dots, x_s^{m_s}]$$

$$E = \bigoplus_{i=1}^{s} E_i$$

$$E_i = \mathrm{Ker}(\mathfrak{m}_i^{\alpha_i})$$

$$m_i = \dim E_i$$

$$\mathfrak{m}_i \mapsto x_i \in \mathbb{k}^r$$

A THEOREM OF WEYL

$$\iota_{n,r} : \mathrm{Chow}_n(\mathbb{A}^r_k) \xhookrightarrow{\text{closed embedding}} \mathbb{A}\left(k^r \oplus \mathrm{Sym}^2 k^r \oplus \dots \oplus \mathrm{Sym}^n k^m\right)$$

$$[x_1, \dots, x_n] \longmapsto (\sigma_1, \dots, \sigma_n)$$

with

$$\sigma_1(x_1, \dots, x_n) = x_1 + \dots + x_n$$

$$\sigma_2(x_1, \dots, x_n) = x_1 x_2 + x_1 x_3 + \dots + x_{n-1} x_n$$

$$\vdots$$

$$\sigma_n(x_1, \dots, x_n) = x_1 \dots x_n$$

[For $r = 1$, this is an isomorphism]

The Universal Spectral Cover

$(x,[x_1,\dots,x_n]) \longmapsto (x-x_1)\cdots(x-x_n) = x^n - \sigma_1 x^{n-1} + \dots + (-1)^n \sigma_n$

$$
\begin{tikzcd}
\mathrm{Cayley}_n(\mathbb{A}^r_k) \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & & \\
\chi_{n,r}^{-1}(0) \arrow[r, hook] \arrow[dr, "p_{n,r}"'] & \mathbb{A}^r_k \times \mathrm{Chow}_n(\mathbb{A}^r_k) \arrow[r, "\chi_{n,r}"] \arrow[d, "pr_2"] & \mathbb{A}(\mathrm{Sym}^n k^r) \\
& \mathrm{Chow}_n(\mathbb{A}^r_k) &
\end{tikzcd}
$$

$p_{n,r}$

UNIVERSAL SPECTRAL COVER

UNIVERSAL CAYLEY - HAMILTON

$$\begin{tikzcd} \mathrm{Cayley}_n(\mathbb{A}^r_k) \arrow[d, "p_{n,r}"'] \\ \mathrm{Chow}_n(\mathbb{A}^r_k) \end{tikzcd}$$

- $p_{n,r}$ is finite of degree n
- $p_{n,r}$ is étale over $\mathrm{Chow}'_n(\mathbb{A}^r_k)$

  MULTIPLICITY FREE TUPLES

- for $a = [x_1^{m_1}, \dots, x_s^{m_s}] \in \mathrm{Chow}_n(\mathbb{A}^r_k)$

$$p_{n,r}^{-1}(a) = \mathrm{Spec}\left(\frac{k[X_1, \dots, X_r]}{\mathfrak{m}_1^{m_1} \cdots \mathfrak{m}_s^{m_s}}\right)$$

- $p_{n,r}$ is flat $\iff$ $r=1$ or $n=1$
- for $A = k[\varphi_1, \dots, \varphi_r]$, $\mathrm{Supp}\, \mathcal{F}_r = p_{n,r}^{-1}(\mathrm{sd}(A))$.

THE HITCHIN MAP

X smooth projective variety $/ \Bbbk$

$$\begin{array}{c} V \\ \downarrow \text{ rank } r \text{ vector bundle} \\ X \end{array}$$

$$\mathcal{M}_{n,V} = \left\{ \begin{array}{ll} \text{pairs } (E,\varphi) \text{ with:} & \\ \begin{array}{c} E \\ \downarrow \\ X \end{array} \begin{array}{l} \text{rank } n \\ \text{vector bundle} \end{array} & \begin{array}{l} \varphi : E \to E \otimes V \\ \varphi \wedge \varphi = 0 \end{array} \end{array} \right\}$$

[Locally, $\varphi = \sum \varphi_i v_i$, $\varphi_i \in H^0(U, \mathrm{End}\, E)$, $[\varphi_i, \varphi_j] = 0$]

When $V = T^*X$, (E,φ) is a <u>Higgs bundle</u> (Hitchin '87, Simpson '88)

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Hitchin map

$$\mathcal{M}_{n,V} \xrightarrow{h_{n,V}} \bigoplus_{i=1}^{n} H^0(X, \mathrm{Sym}^i V)$$

$$(E,\varphi) \longmapsto (\sigma_1(\varphi), \dots, \sigma_n(\varphi))$$

where,

$$p_\varphi(T) = \det(T - \varphi) = T^n + \sum_{i=1}^{n} \sigma_i(\varphi) T^{n-1}$$

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PROBLEM: Study the fibres of $h_{n,V}$.

The Hitchin Map $\left(\begin{array}{l}\dim X = 1 \\ V = L \text{ line bundle}\end{array}\right)$

$$h_{n,L} : \mathcal{M}_{n,L} \longrightarrow \bigoplus_{i=1}^{n} H^0(X, L^{\otimes i})$$

<u>Thm</u> (Hitchin '87, Beauville-Narasimhan-Ramanan '89)

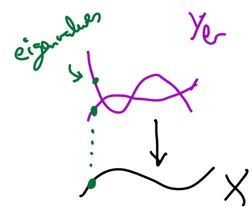
Assume $L^{\otimes n}$ is base point free.

For a generic $b \in \bigoplus_{i=1}^{n} H^0(X, L^{\otimes i})$

$\exists \quad Y_b \xrightarrow[\text{of deg. } n]{\text{finite}} X$ s.t.

Spectral curve

smooth



$$\boxed{h_{n,L}^{-1}(b) \cong \mathrm{Pic}(Y_b)}.$$

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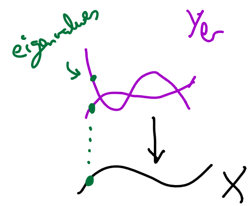

We want similar results for general $\dim X$ and $\mathrm{rk}\, V$.

Factorization of the Hitchin map

$$\begin{array}{c} \mathrm{Chow}_n(V/X) \\ \downarrow \\ X \end{array} \leftarrow \text{bundle with fibres } \mathrm{Chow}_n(V_x)$$

$$\rightsquigarrow \quad \mathcal{B}_{n,V} = \Gamma\left(\mathrm{Chow}_n(V/X)\right)$$

$$\mathrm{Chow}_n(\mathbb{A}^r_k) \xhookrightarrow{\iota_{n,r}} \mathbb{A}\left(k^r \oplus \mathrm{Sym}^2 k^r \oplus \dots \oplus \mathrm{Sym}^n k^r\right)$$

$$\rightsquigarrow$$

$$\mathcal{B}_{n,V} \xhookrightarrow[\iota_{n,V}]{} \bigoplus_{i=1}^{n} H^0(X, \mathrm{Sym}^i V)$$

FACTORIZATION OF THE HITCHIN MAP

$$\begin{tikzcd} \mathrm{Chow}_n(V/X) \arrow[d] \\ X \end{tikzcd} \quad \leftarrow \text{bundle with fibres } \mathrm{Chow}_n(V_x)$$

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 & \mathcal{B}_{n,V} \arrow[r, hook, "\iota_{n,V}"'] & \bigoplus_{i=1}^{n} H^0(X, \mathrm{Sym}^i V) \\
\mathcal{M}_{n,V} \arrow[ur, "\mathrm{sd}"'] \arrow[urr, bend right, "h_{n,V}"'] & &
\end{tikzcd}$$

$$(E,\varphi) \longmapsto \left[x \mapsto \mathrm{sd}(\varphi(x))\right]$$

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$$(E,\varphi) \mapsto \left[x \mapsto sd(\varphi(x))\right]$$

$$\boxed{\text{STUDY FIBRES OF } sd}$$

THE SPECTRAL COVER

$b \in \mathcal{B}_{n,V}$

$$\begin{tikzcd} Y_b \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\text{spectral cover}"] & \mathrm{Cayley}_n(V/X) \arrow[d, "p_{n,V}"] \\ X \arrow[r, "b"'] & \mathrm{Chow}_n(V/X) \end{tikzcd}$$

π is:

- finite of deg. n
- generically étale if
  $b(\text{generic point of } X) \in \mathrm{Chow}'(V/X)$
- NOT FLAT IN GENERAL!!
- $$\begin{tikzcd} Y_b \arrow[r, hook] \arrow[dr, "\pi"'] & V \arrow[d, "p"] \\ & X \end{tikzcd}$$

THE SPECTRAL CORRESPONDENCE

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Suppose $(E,\varphi) \in sd^{-1}(b)$

$$\varphi : E \longrightarrow E \otimes V$$

$$\rightsquigarrow$$

$$V^\vee \longrightarrow \mathrm{End}(E)$$

$$\begin{tikzcd}
 & V \arrow[r, "p"] & X \\
\mathcal{F} \arrow[ur, squiggly] \arrow[rr, squiggly] & & E = p_* \mathcal{F} \arrow[u]
\end{tikzcd}$$

Universal Cayley-Hamilton $\Rightarrow$ $\mathrm{Supp}\, \mathcal{F} \subset Y_b$

$$\left\{ \begin{array}{c} (E,\varphi) \text{ with} \\ sd(E,\varphi) = b \end{array} \right\} = \left\{ \begin{array}{l} \text{qcoh. sheaves } \mathcal{L} \text{ on } Y_b \\ \text{with } \pi_* \mathcal{L} = E \end{array} \right\}$$

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Lemma A regular ring. $A \xrightarrow{\text{finite}} B$

$$\left\{ \begin{array}{c} \text{Locally free} \\ A\text{-modules} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Maximal Cohen-Macaulay} \\ B\text{-modules} \end{array} \right\}$$

THE SPECTRAL CORRESPONDENCE

$b \in \mathcal{B}_{n,V}$

$$\begin{tikzcd} Y_b \arrow[r] \arrow[d, "\pi"'] \arrow[d, phantom, "\text{spectral cover}", shift left=6ex] & \mathrm{Cayley}_n(V/X) \arrow[d, "p_{n,V}"] \\ X \arrow[r, "b"'] & \mathrm{Chow}_n(V/X) \end{tikzcd}$$

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$$\boxed{sd^{-1}(b) = \left\{ \begin{matrix} \text{Maximal Cohen-Macaulay} \\ \text{sheaves on } Y_b \\ \text{of generic rk. } 1 \end{matrix} \right\}}$$

Cohen Macaulayfications

$$sd^{-1}(\mathfrak{b}) = \underbrace{\left\{ \begin{array}{l} \text{Maximal Cohen-Macaulay} \\ \text{sheaves on } Y_{\mathfrak{b}} \\ \text{of generic rk. } 1 \end{array} \right\}}_{\text{THIS MIGHT BE EMPTY}}$$

SOLUTION: Obtain Cohenmacaulayfication

$$\begin{tikzcd} & Y_{\mathfrak{b}} \arrow[d] \\ \tilde{Y}_{\mathfrak{b}} \arrow[ur] \arrow[r] & X \end{tikzcd}$$

- For $\dim X = 1$: $\pi_* \mathcal{O}_{Y_{\mathfrak{b}}} = \underset{\text{locally free}}{F} \oplus \underset{\text{torsion}}{T}$, $\quad \tilde{Y}_{\mathfrak{b}} = \underline{\mathrm{Spec}}(F)$.
- For $\dim X = 2$: [Chen-Ngo, 2020]
- For $\dim X > 2$: ???

Open Questions

- Conditions for $\tilde{Y}_e$ to be <u>integral</u> or <u>smooth</u>

  [when $\mathrm{rk}\, V = 1 \rightsquigarrow$ Bertini ($\tilde{Y}_e$ is a divisor)]

- In $\dim X = 1$, if $\det V = T^*X$, V is noncompact Calabi-Yau 3-fold.

  $\rightsquigarrow$ Interesting geometry?

  [Xie-Yonekura, 2014 (physicists)]

- $GL_n \overset{?}{\rightsquigarrow} G$ split reductive group

  [Chen-Ngo, 2020]

  [Problem: How to generalize sd?]

THANKS

FOR YOUR

ATTENTION

~ QUESTIONS ?