• Aplicaciones de la teoría de Hodge a superficies de Riemann

<u>Teorema</u> (Descomposición de Hodge)

$$\underbrace{\mathbb{C}^{2g} \cong H^1(M,\mathbb{C})}_{\text{(de Riemann)}} \cong H^{1,0}(M) \oplus H^{0,1}(M)$$

[Por tanto, como $H^{1,0}(M) \cong H^{0,1}(M)$.

$$H^1(M,\mathcal{O}) \cong H^{0,1}(M) \cong \mathbb{C}^g.$$

↑ ANALÍTICO (ALGEBRAICO) ↑ Dolbeault ↑ TOPOLÓGICO]

$\left(\dim H^1(M,\mathcal{O}) = g\right)$ ← Así se define el género en GEOALG.

<u>-Dem.</u>

$$\Omega^1(M,\mathbb{C}) \cong \Omega^{1,0}(M) \oplus \Omega^{0,1}(M) \cong \mathcal{H}^{1,0}(M) \oplus \mathcal{H}^{0,1}(M) \oplus \bar\partial^*\big(\overbrace{G\bar\partial\Omega^{1,0}(M)}^{\Omega^{1,1}(M)}\big) \oplus \bar\partial^*\big(\underbrace{G\bar\partial\Omega^{0,1}(M)}_{\Omega^{0,2}(M)=0}\big) \oplus \bar\partial\big(\underbrace{G\bar\partial^*\Omega^{1,0}(M)}_{\cancel{\Omega^{1,-1}(M)}=0}\big) \oplus \bar\partial\big(\underbrace{G\bar\partial^*\Omega^{0,1}(M)}_{\Omega^{0,0}(M)\cong C^\infty(M,\mathbb{C})}\big)$$

$$\cong \mathcal{H}^{1,0}(M) \oplus \mathcal{H}^{0,1}(M) \oplus \underbrace{\bar\partial^*(\Omega^{1,1}(M))}_{*\bar\partial\underbrace{*\Omega^{1,1}(M)}_{\Omega^{0,0}(M)\cong C^\infty(M,\mathbb{C})}} \oplus \bar\partial C^\infty(M,\mathbb{C})$$

$$*\bar\partial(f) = *\left(\frac{\partial f}{\partial \bar z} d\bar z\right) = \overline{\frac{\partial f}{\partial \bar z}}\, dz = \frac{\partial \bar f}{\partial z}\, dz \qquad \Longrightarrow *\bar\partial C^\infty(M,\mathbb{C}) = \partial C^\infty(M,\mathbb{C})$$

[$\alpha \wedge *\beta = \langle \alpha,\beta\rangle\, d\mathrm{vol}$; $\quad a\,dz \wedge *(b\,dz) = a\bar b\, dz\wedge d\bar z \Rightarrow *\beta = \bar b\, d\bar z$]

[$\overline{\frac{\partial f}{\partial \bar z}} = \frac{\partial \bar f}{\partial z}$ Ej. Comprobarlo [Escribir $f = u+iv$ y $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$]]

$$\Omega^1(M,\mathbb{C}) \cong \mathcal{H}^{1,0}(M) \oplus \mathcal{H}^{0,1}(M) \oplus \overbrace{\partial C^\infty(M,\mathbb{C}) \oplus \bar\partial C^\infty(M,\mathbb{C})}^{dC^\infty(M,\mathbb{C})}$$

$$H^1(M,\mathbb{C}) = \frac{\Omega^1(M,\mathbb{C})}{dC^\infty(M,\mathbb{C})} \cong \mathcal{H}^{1,0}(M) \oplus \mathcal{H}^{0,1}(M) \cong H^{1,0}(M) \oplus H^{0,1}(M). \ \#$$

• Grupo de Picard

$\boxed{\text{Holomorfas } U \to \mathbb{C}^*}$

(Funciones de transición)

Clasifica los fibrados de línea holomorfos: $\mathrm{Pic}(M) = H^1(M, \mathcal{O}^*)$

Sucesión exacta exponencial:

$$0 \to \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$
$$f \longmapsto e^{2\pi i f}$$

Sucesión exacta larga en cohomología:

$$\cdots \to H^0(M,\mathbb{Z}) \to H^0(M,\mathcal{O}) \to H^0(M,\mathcal{O}^*) \to H^1(M,\mathbb{Z}) \to H^1(M,\mathcal{O}) \to H^1(M,\mathcal{O}^*) \xrightarrow{c_1} H^2(M,\mathbb{Z}) \to \cdots$$

c_1 ← Es la clase de Chern.

En el caso M sup. Riemann, $H^2(M,\mathbb{Z}) \cong \mathbb{Z}$ y

$$c_1 : \underset{\mathrm{Pic}(M)}{\overset{H^1(M,\mathcal{O}^*)}{\|}} \longrightarrow \mathbb{Z}$$
$$L \longmapsto \deg L$$

da el grado del fibrado de línea.

Llamamos $\mathrm{Pic}^0(M) = \mathrm{Ker}\, c_1$ ← Fibrados de línea de grado 0.

Además $H^1(M,\mathbb{Z}) \hookrightarrow H^1(M,\mathcal{O})$. Por tanto:

$$0 \to H^1(M,\mathbb{Z}) \to H^1(M,\mathcal{O}) \to \mathrm{Pic}_0(M) \to 0 \text{ es exacta,}$$

luego $\mathrm{Pic}_0(M) \cong \dfrac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})} \cong \dfrac{\mathbb{C}^g}{\mathbb{Z}^{2g}} \cong \boxed{\mathbb{T}^{2g}}$.

$\dfrac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})} = \mathrm{Jac}(M)$ ← $\boxed{\text{Jacobiana de } M}$

OBS: $\mathrm{Pic}_d(M) \xrightarrow{\cong} \mathrm{Jac}(M)$, $L \longmapsto L \otimes M$, $M \in \mathrm{Pic}_{-d}(M)$.