Higgs bundles twisted by a vector bundle

V-twisted Higgs bundles

X — Riemann surface of genus g.
$V \to X$ — vector bundle of rank 2.
V-twisted Higgs bundle — a pair (E, φ), with $E \to X$ a vector bundle and $\varphi : E \to E \otimes V$ with $\varphi \wedge \varphi = 0$.

$\mathrm{SL}(2, \mathbb{C})$ pairs

Fix the structure group to $\mathrm{SL}(2, \mathbb{C})$ by imposing

- $\operatorname{rk} E = 2$,
- $\det E = \mathcal{O}_X$,
- $\operatorname{tr} \varphi = 0$.

$\mathcal{M}$ — **moduli space** of V-twisted $\mathrm{SL}(2, \mathbb{C})$-Higgs bundles.

The Hitchin map

The **Hitchin map** is defined as

$$h : \mathcal{M} \longrightarrow H^0(X, \mathrm{Sym}^2 V)$$
$$(E, \varphi) \longmapsto \operatorname{tr}(\varphi^2).$$

It is some sort of "characteristic polynomial"

$$\det(\varphi - \mathrm{id}_E \otimes T) = T^2 - \operatorname{tr}(\varphi^2).$$

Its image lies inside the **Hitchin base** $\mathcal{B} \subset H^0(X, \mathrm{Sym}^2 V)$ defined as

$$\mathcal{B} = \left\{ b = (b_1, b_2, b_3) \text{ locally}, \ b_3^2 = b_1 b_2 \right\}.$$

We are interested in studying the fibres of this map.

Related objects

V-twisted Higgs bundles over curves are halfway between the theories of Higgs bundles over curves (Hitchin, 1987) and over higher dimensional varieties (Simpson, 1988). Higgs bundles are pairs (E, φ) as above, with $V = T^*X$. They are related to representations of the fundamental group through **nonabelian Hodge theory** (Corlette 1988 and Simpson 1988).
The Hitchin map was introduced by Hitchin (1987) for Higgs bundles over curves and later studied by Beauville–Narasimhan–Ramanan (1989) for a general twisting line bundle. Over higher dimensional varieties, the Hitchin map was defined by Simpson (1994) and recently studied by Chen–Ngo (2020).

Universal spectral data

E — $\mathbb{C}$-vector space of dimension 2
Commuting variety:

$$\mathfrak{C} = \{(\varphi_1, \varphi_2) \in \mathrm{End}_0 E : [\varphi_1, \varphi_2] = 0\}.$$

"Point model" of the Hitchin map:

$$h : \mathfrak{C} \longrightarrow \mathbb{C}^3 \cong \mathrm{Sym}^2 E$$
$$(\varphi_1, \varphi_2) \longmapsto (\operatorname{tr}(\varphi_1^2), \operatorname{tr}(\varphi_2^2), \operatorname{tr}(\varphi_1\varphi_2)).$$

v — common eigenvector of φ_1 and φ_2
λ_1, λ_2 — associated eigenvalues
Choice of the eigenvector induces an action of $\mathbb{Z}_2$ on $\mathbb{C}^2$ as

$$-1 \cdot (\lambda_1, \lambda_2) = (-\lambda_1, -\lambda_2).$$

Universal spectral data map:

$$\mathrm{sd} : \mathfrak{C} \longrightarrow \mathbb{C}^2/\mathbb{Z}_2$$
$$(\varphi_1, \varphi_2) \longmapsto [(\lambda_1, \lambda_2)].$$

The quotient $\mathbb{C}^2/\mathbb{Z}_2$ can be identified with the cone

$$B = \left\{ (x, y, z) \in \mathbb{C}^3 : z^2 = xy \right\},$$

through the map

$$\iota : \mathbb{C}^2/\mathbb{Z}_2 \longrightarrow \mathbb{C}^3$$
$$[(\lambda_1, \lambda_2)] \longmapsto (\lambda_1^2, \lambda_2^2, \lambda_1\lambda_2).$$

There is an obvious factorization $h = \iota \circ \mathrm{sd}$.

Universal characteristic polynomial:

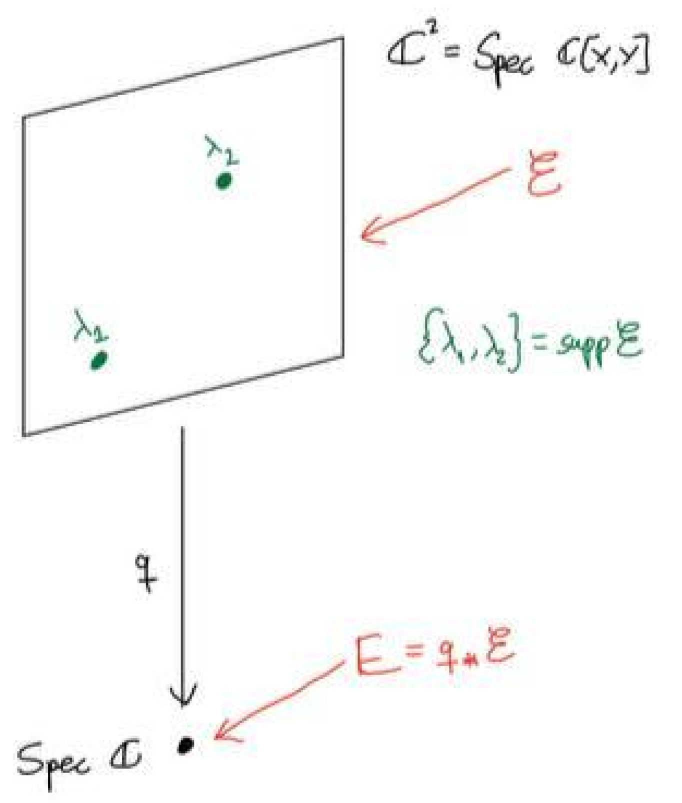
$$\chi : \mathbb{C}^2 \times B \longrightarrow \mathbb{C}^3$$
$$((x, y), (b_1, b_2, b_3)) \longmapsto (x^2 - b_1, y^2 - b_2, xy - b_3).$$

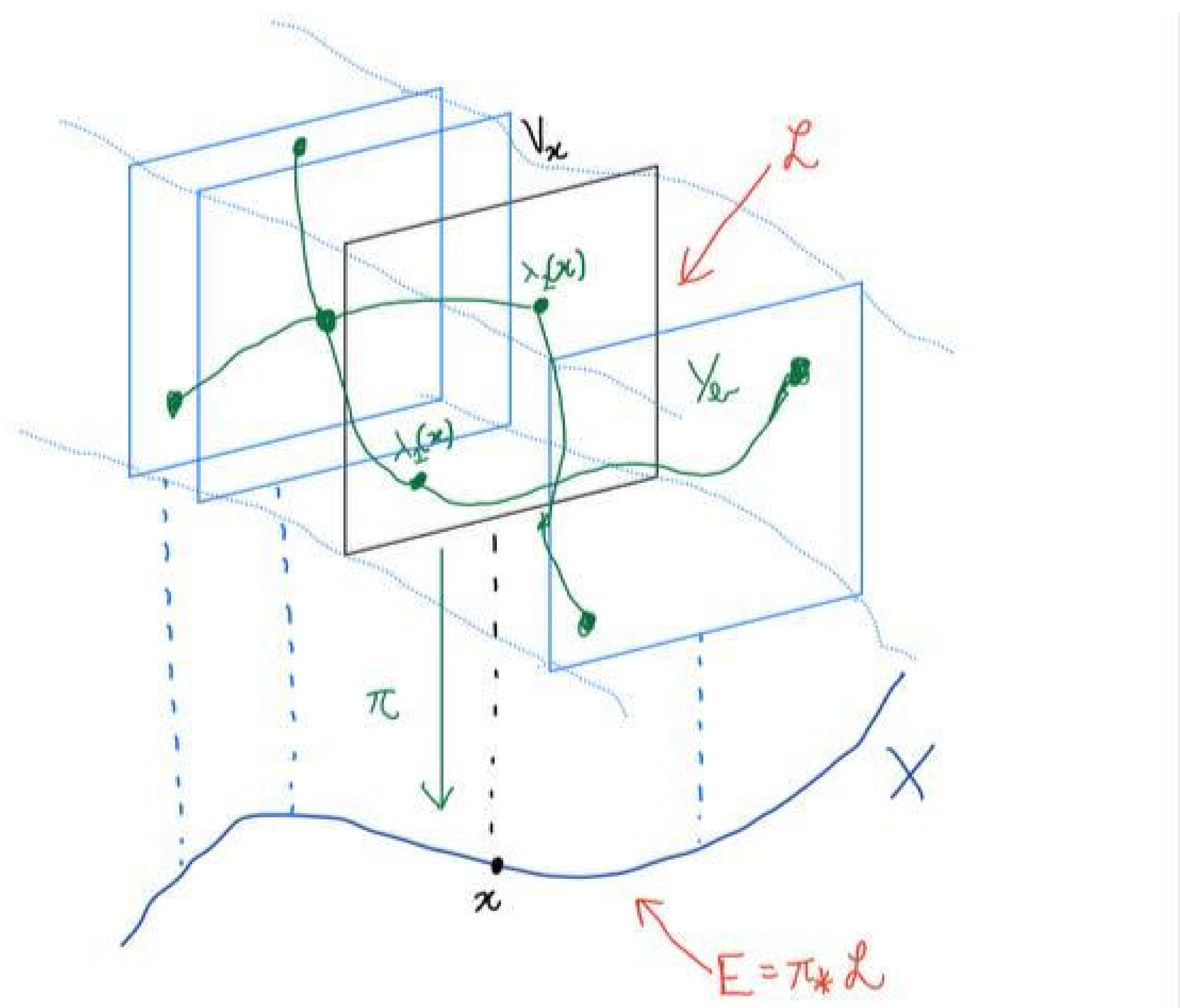
Universal spectral cover: the natural projection

$$p : \chi^{-1}(0) \longrightarrow B.$$

Theorem ("Cayley–Hamilton")

Let $\mathcal{E}$ be the $\mathbb{C}[X, Y]$-module structure induced on E by the map $\mathbb{C}[X, Y] \to \mathrm{End}\, E$ given by evaluation on (φ_1, φ_2). The support of $\mathcal{E}$ is contained in $p^{-1}(h(\varphi_1, \varphi_2))$.





The spectral curve

Note that $\mathcal{B}$ = sections of $V \times_{\mathrm{GL}(2,\mathbb{C})} B$.
Given $b \in \mathcal{B}$ we define the **spectral curve** associated to b

$$\begin{array}{ccc} Y_b & \longrightarrow & V \times_{\mathrm{GL}(2,\mathbb{C})} \chi^{-1}(0) \\ \downarrow \pi & & \downarrow \\ X & \xrightarrow{\ b\ } & V \times_{\mathrm{GL}(2,\mathbb{C})} B. \end{array}$$

The spectral curve is not flat in general. We can obtain a **flat modification** $\tilde{\pi} : \tilde{Y}_b \to X$ by removing torsion.

Theorem (The spectral correspondence)

For any $b \in \mathcal{B}$ generically nonzero, π_* gives an equivalence of categories between V-twisted Higgs bundles with $h(\varphi) = b$ and coherent sheaves $\mathcal{L}$ on Y_b of rank 1 with $\pi_*\mathcal{L}$ locally free of trivial determinant. In particular, this category is non empty.
Moreover, if Y_b is irreducible, the corresponding sheaves on $\tilde{Y}_b$ are torsion free and, if $\tilde{Y}_b$ is smooth of genus g' the fibre is an abelian variety of dimension $g - g'$.

Properties of the spectral curve

We are studying under what conditions on b the spectral curve will have good properties, like being integral or smooth.

Theorem (G–García-Prada–Narasimhan, 2021)

1. *If the map $H^0(X, V) \to \mathcal{B}$ given by $a \mapsto a^2$ is not surjective, then for a generic $b \in \mathcal{B}$ the flat spectral curve $\tilde{Y}_b$ is integral.*
2. *If b is a nonvanishing section, then π is étale. In particular, Y_b has genus $2g - 1$ and $h^{-1}(b)$ has dimension $g - 1$.*
3. *If the flat spectral curve $\tilde{Y}_b$ is irreducible, then it is smooth if and only if b does not have a multiple zero.*

Main reference

G, Oscar García-Prada & M.S. Narasimhan (2021). "Higgs bundles twisted by a vector bundle". Arxiv preprint: 2105.05543

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