Multiplicative Higgs bundles and involutions

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Contents

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- k — algebraically closed field of char. 0
- G — semisimple simply-connected algebraic group over k, $B \subset G$ Borel subgroup, $T \subset B$ maximal torus
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- $d \in \mathbb{N}$, $X_d = X^d / \mathfrak{S}_d$ (elements are effective divisors of deg. d)
- $n \in \mathbb{N}$, $\boldsymbol{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, $X_{\boldsymbol{d}} = X_{d_1} \times \cdots \times X_{d_n}$.
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Stack of multiplicative G-Higgs bundles

$$\mathcal{M}_{\boldsymbol{d}}(G) = \left\{(\boldsymbol{D}, E, \varphi) : \boldsymbol{D} \in X_{\boldsymbol{d}}, E \in \mathrm{Bun}_G(X), \varphi \in \Gamma(X \setminus |D|, E(G))\right\}.$$

The invariant

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$G(\mathcal{O})\backslash G(K)/G(\mathcal{O}) \cong \mathrm{X}_*(T)_+$.

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- $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathrm{X}_*(T)_+)^n$.

Stack of multiplicative G-Higgs bundles of type $\boldsymbol{\lambda}$

$\mathcal{M}_{\boldsymbol{d},\boldsymbol{\lambda}}(G) = \{(\boldsymbol{D}, E, \varphi) \in \mathcal{M}_{\boldsymbol{d}}(G) : \mathrm{inv}(\varphi) \leq \boldsymbol{\lambda} \cdot \boldsymbol{D}\}$.

Multiplicative Chevalley restriction

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Multiplicative Hitchin map

Let $\mathcal{B}_{\boldsymbol{d},\boldsymbol{\lambda}}(G) \to X_{\boldsymbol{d}}$ with $\mathcal{B}_{\boldsymbol{d},\boldsymbol{\lambda}}(G)_{\boldsymbol{D}} := \bigoplus_{i=1}^{r} H^0(X, \mathcal{O}_X(\langle \omega_i, \boldsymbol{\lambda} \cdot \boldsymbol{D} \rangle))$.

$$\begin{aligned} h_{\boldsymbol{d},\boldsymbol{\lambda}} : \mathcal{M}_{\boldsymbol{d},\boldsymbol{\lambda}}(G) &\longrightarrow \mathcal{B}_{\boldsymbol{d},\boldsymbol{\lambda}}(G) \\ (\boldsymbol{D}, E, \varphi) &\longmapsto (b_1(\varphi), \ldots, b_r(\varphi)) \end{aligned}$$

The multiplicative Hitchin map. Some history

- Present in the physics literature since the late 90s.
- 2002. Introduced in the AG literature by Hurtubise–Markman (integrable system).
- 2010. HK-style correspondence of multiplicative $\mathrm{GL}_r(\mathbb{C})$-Higgs bundles with "Hermitian–Einstein" singular $U(r)$-monopoles, by Charbonneau–Hurtubise. Extended by Smith (2016) and Mochizuki (2017).
- 2011. Considered by Frenkel–Ngô in the context of geometrization of trace formulas. They suggest the Vinberg monoid approach, further developed in the works of Bouthier (2014-15) and J. Chi (2018).
- 2023. Used in the proof of the Fundamental Lemma for the groups in the thesis of G. Wang.

Contents

Reductive monoids

- A monoid M is a semigroup (binary operation with associativity) with identity element. Invertible elements form a group $M^\times$.
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- Example:

$$\begin{aligned}\mathrm{Mat}_{n\times n} : \mathrm{Alg}_k &\longrightarrow \mathrm{Monoids} \\ A &\longmapsto \mathrm{Mat}_{n\times n}(A)\end{aligned}$$

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  is a reductive monoid over k with $(\mathrm{Mat}_{n\times n})^\times = \mathrm{GL}_n$.
- An algebraic monoid can also be thought as a $(M^\times \times M^\times)$-equivariant open embedding of its unit group $M^\times$. ("Nonabelian toric varieties").

Abelianization

- G as before. M a reductive monoid with $(M^\times)' = G$.
- The GIT quotient
$$\alpha_M : M \longrightarrow \mathbf{A}_M := M /\!\!/ (G \times G)$$
is called the abelianization of M.
- $\mathbf{A}_M$ is a toric variety for the torus $Z^0_{M^\times}/(Z^0_{M^\times} \cap G)$.

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- A homomorphism $f\colon M_1 \to M_2$ induces a commutative square
$$\begin{array}{ccc} M_1 & \xrightarrow{\ f\ } & M_2 \\ \big\downarrow{\scriptstyle \alpha_{M_1}} & & \big\downarrow{\scriptstyle \alpha_{M_2}} \\ \mathbf{A}_{M_1} & \longrightarrow & \mathbf{A}_{M_2}. \end{array}$$
- f is excellent if the above square is Cartesian.

Very flat monoids

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The enveloping monoid

The category of very flat reductive monoids M with $(M^\times)' = G$ and excellent morphisms has a versal object $\mathrm{Env}(G)$, called the (Vinberg) enveloping monoid of G.

The enveloping monoid

- G as before. $G_+ := (T \times G)/Z_G$.
- $\omega_1, \dots, \omega_r$ fundamental dominant weights, $\alpha_1, \dots, \alpha_r$ simple roots.
- $\mathrm{Env}(G)$ is defined as the closure of the image of

$$\begin{aligned} G_+ &\longrightarrow \bigoplus_{i=1}^{r} (\mathrm{End}(V_i) \times \mathbb{A}^1) \\ [t, g] &\longmapsto (t^{w_0(\omega_i)} \rho_i(g), t^{\alpha_i})_{i=1}^{r}, \end{aligned}$$

for $\rho_i : G \to \mathrm{GL}(V_i)$ the irrep with highest weight ω_i and w_0 the longest element of W.

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  for $\rho_i : G \to \mathsf{GL}(V_i)$ the irrep with highest weight ω_i and w_0 the longest element of W.
- $Z^0_{M^\times}/(Z^0_{M^\times} \cap G) = T/Z_G = T^{\mathrm{ad}}$.
- $\mathbf{A}_{\mathsf{Env}(G)} = \mathsf{Spec}(k[e^{\alpha_i} : i = 1, \ldots, r]) \cong \mathbb{A}^r$. ($\mathsf{X}^*(T^{\mathrm{ad}})$ = root lattice).
- $\alpha_{\mathsf{Env}(G)}([t, g]) = (t^{\alpha_1}, \ldots, t^{\alpha_r})$.

The multiplicative Hitchin map of a very flat monoid

Invariant theory for the monoid

G as before. M very flat reductive monoid with $(M^\times)' = G$.

$$M /\!\!/ G = (G /\!\!/ G) \times \mathbf{A}_M.$$

The multiplicative Hitchin map of a very flat monoid

Invariant theory for the monoid

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The multiplicative Hitchin map associated to M

Let X as before. We obtain a Hitchin-type fibration

$$\mathcal{M}_X(M) \xrightarrow{h_M} \mathcal{B}_X(M) \longrightarrow \mathcal{A}_X(M) \longrightarrow \mathrm{Bun}_{Z_{M^\times}}(X)$$

by applying the functor $\mathrm{Map}(X, -)$ to the natural sequence of stacky quotients

$$[M/(G \times Z_{M^\times})] \longrightarrow [(M /\!\!/ G)/Z_{M^\times}] \longrightarrow [\mathbf{A}_M/Z_{M^\times}] \longrightarrow \mathbb{B}Z_{M^\times}.$$

Comparing the two pictures

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathsf{X}_*(T)_+)^n \subset (\mathsf{X}_*(T^{\mathrm{ad}})_+)^n$. This defines

$$\begin{aligned} \boldsymbol{\lambda} : \mathbb{G}_m^n &\longrightarrow T^{\mathrm{ad}} \\ (z_1, \ldots, z_n) &\longmapsto z_1^{\lambda_1} \cdots z_n^{\lambda_n}, \end{aligned}$$

which extends to a map $\boldsymbol{\lambda} : \mathbb{A}^n \to \mathbf{A}_{\mathsf{Env}(G)}$.

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which extends to a map $\boldsymbol{\lambda} : \mathbb{A}^n \to \mathbf{A}_{\mathsf{Env}(G)}$.
Consider $M_{\boldsymbol{\lambda}} = \mathsf{Env}(G) \times_{\boldsymbol{\lambda}} \mathbb{A}^n$ the corresponding very flat monoid. Note that $\mathbf{A}_{M_{\boldsymbol{\lambda}}} = \mathbb{A}^n$, so $\mathbb{B}Z_{M_{\boldsymbol{\lambda}}^\times} = \mathsf{Pic}(X)^n$, and for any tuple of line bundles $\boldsymbol{L} = (L_1, \dots, L_n)$,

$$\mathcal{A}_X(M_{\boldsymbol{\lambda}})_{\boldsymbol{L}} = \bigoplus_{i=1}^n H^0(X, L_i).$$

Comparing the two pictures

Let $\boldsymbol{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ and $\boldsymbol{D} = (D_1, \ldots, D_n) \in X_{\boldsymbol{d}}$.
Since each D_i is effective, there exists a canonical section s_i of each $\mathcal{O}_X(D_i)$. Let us denote $\mathcal{O}_X(\boldsymbol{D}) = \bigoplus_{i=1}^n \mathcal{O}_X(D_i)$ and $\boldsymbol{s} = (s_1, \ldots, s_n)$.

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Theorem (Bouthier, J. Chi, G. Wang)

The map $X_{\boldsymbol{d}} \to \mathcal{A}_X(M_{\boldsymbol{\lambda}})$, $\boldsymbol{D} \mapsto (\mathcal{O}_X(\boldsymbol{D}), \boldsymbol{s})$ induces the following diagram, where all squares are Cartesian

$$\begin{array}{ccccc} \mathcal{M}_{\boldsymbol{d},\boldsymbol{\lambda}}(G) & \xrightarrow{h_{\boldsymbol{d},\boldsymbol{\lambda}}} & \mathcal{B}_{\boldsymbol{d},\boldsymbol{\lambda}}(G) & \longrightarrow & X_{\boldsymbol{d}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_X(M_{\boldsymbol{\lambda}}) & \xrightarrow{h_{M_{\boldsymbol{\lambda}}}} & \mathcal{B}_X(M_{\boldsymbol{\lambda}}) & \longrightarrow & \mathcal{A}_X(M_{\boldsymbol{\lambda}}). \end{array}$$

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(Additive) Higgs bundles and involutions

- G as before. An involution θ of G is an order 2 automorphism of G.
- $G^\theta := \{g \in G : \theta(g) = g\}$, $G^\theta_0 = (G^\theta)^0$, $G_\theta := \left\{g \in G : \theta(g)g^{-1} \in Z_G\right\} = N_G(G^\theta)$.

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Stack of (G, θ)-Higgs bundles

$$\mathrm{Higgs}_X(G, \theta) = \left\{(E, \varphi) : E \in \mathrm{Bun}_{G^\theta}(X), \varphi \in H^0(X, E(\mathfrak{m}) \otimes K_X)\right\}.$$

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- Our motivation: Study the multiplicative analogue.

(Additive) Higgs bundles and involutions. Motivation

Stack of (G, θ)-Higgs bundles

$$\mathsf{Higgs}_X(G,\theta) = \left\{(E,\varphi) : E \in \mathsf{Bun}_{G^\theta}(X), \varphi \in H^0(X, E(\mathfrak{m}) \otimes K_X)\right\}.$$

- (For $k = \mathbb{C}$). Under the nonabelian Hodge correspondence, (polystable) (G,θ)-Higgs bundles yield representations of $\pi_1(X^{\mathrm{an}})$ on the real form $G_{\mathbb{R}}$ of G determined by θ.
- $\mathsf{Higgs}_X(G,\theta)$ appears as fixed points of

$$\begin{aligned} \mathsf{Higgs}_X(G) &\longrightarrow \mathsf{Higgs}_X(G) \\ (E,\varphi) &\longmapsto (\theta(E), -\theta(\varphi)). \end{aligned}$$

- $\mathsf{Higgs}_X(G,\theta)$ is the support of a BAA-brane of $\mathsf{Higgs}_X(G)$, conjecturally mirror to the BBB-brane inside $\mathsf{Higgs}_X({}^L G)$ given by the Nadler dual group of (G,θ).

Contents

Root data of (G, θ)

- (G, θ) as before. A torus $A \subset G$ is θ-split if $\theta(a) = a^{-1}$ for all $a \in A$.
- $A \subset G$ maximal θ-split. If $A \subset T$ maximal torus, T is θ-stable ($\theta(T) \subset T$).

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- θ acts on the roots Φ_T.
- If $\Phi_T^\theta = \varnothing$ we say that θ is quasi-split.

Restricted root system

$$\Phi_\theta := \left\{ \bar{\alpha} = \frac{\alpha - \theta(\alpha)}{2} \in \mathrm{X}^*(A) \otimes \mathbb{R} : \alpha \in \Phi_T \setminus \Phi_T^\theta \right\}.$$

This is a (possibly non-reduced) root system in $\mathrm{X}^*(A) \otimes \mathbb{R}$ with Weyl group $W_\theta = N_G(A)/Z_G(A)$ the little Weyl group.

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- Root lattice: $\mathrm{R}(\Phi_\theta) = \mathrm{X}^*(A/(A \cap G_\theta))$.
- Weight lattice: $\mathrm{P}(\Phi_\theta) = \mathrm{X}^*(A/(A \cap G^\theta)) = \mathbb{Z}\langle \varpi_1, \ldots, \varpi_l \rangle$.

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"Cartan decomposition" (Uzawa, Luna–Vust, Nadler)

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"Chevalley restriction" (Richardson)

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Moreover, there are G^θ-invariant functions $b_1, \ldots, b_l$ with highest weights $\varpi_1, \ldots, \varpi_l$ such that

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- Let G_1 be any reductive group.
- A symmetric G_1-variety is an algebraic homogeneous space of the form G_1/H_1 with

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for some involution $\vartheta \in \mathrm{Aut}_2(G_1)$.

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for some involution $\vartheta \in \mathrm{Aut}_2(G_1)$.
- Any symmetric G_1-variety is of the form $(G_1' \times Z)/H$, for some torus Z and $(G_1' \times Z)_0^{\vartheta} \subset H \subset (G_1' \times Z)_{\vartheta}$, where

$$\begin{aligned} \vartheta : G_1' \times Z &\longrightarrow G_1' \times Z \\ (g, z) &\longmapsto (\theta(g), z^{-1}), \end{aligned}$$

for $\theta \in \mathrm{Aut}_2(G_1')$. The symmetric G_1'-variety $G_1'/(H \cap (G_1' \times \{1\}))$ is called the semisimple part of $(G_1' \times Z)/H$.

Symmetric embeddings

- A symmetric G_1-embedding is a normal G_1-variety Σ with a G_1-equivariant Zariski open embedding $O_\Sigma \hookrightarrow \Sigma$, where O_Σ is a symmetric G_1-variety. Σ is simple if it has only one closed G_1-orbit.

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- Recall that O_Σ is of the form $(G_1' \times Z)/H$. We have the following tori

$$Z_\Sigma = Z/\left\{z \in Z : z^2 = 1\right\} \quad \text{and} \quad A_\Sigma = Z/\mathrm{pr}_2(H).$$

- The abelianization of Σ is the GIT quotient $\alpha_\Sigma : \Sigma \to \mathbf{A}_\Sigma := \Sigma /\!\!/ G_1'$.
- $\mathbf{A}_\Sigma$ is a toric variety for the torus A_Σ.
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- (G, θ), $T \subset G$ and $A \subset T$ as before.

The enveloping embedding

The category of very flat symmetric embeddings Σ such that the semisimple part of O_Σ is G/G^θ and excellent morphisms has a versal object $\mathrm{Env}(G/G^\theta)$, called the (Guay) enveloping embedding of G/G^θ.

The enveloping embedding

- $(G/G^\theta)_+ = (A \times G)/\left\{(an^{-1}, nh) : h \in G^\theta, a \in A \cap G^\theta, n \in A \cap G_\theta\right\}$.
- $\mathrm{Env}(G/G^\theta)$ is the closure of the image of

$$\begin{aligned}(G/G^\theta)_+ &\longrightarrow \bigoplus_{i=1}^{l} (\mathbb{A}^{m_i} \times \mathbb{A}^1)\\ [a,g] &\longmapsto (a^{w_0(\varpi_i)}(f_i^1(g), \dots, f_i^{m_i}(g)), a^{-\bar{\alpha}_i})_{i=1}^{l},\end{aligned}$$

where $f_i^1, \dots, f_i^{m_i}$ is a basis of the G-submodule $k[G/G^\theta]_{\varpi_i}$ as a k-vector space.

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  where $f_i^1, \dots, f_i^{m_i}$ is a basis of the G-submodule $k[G/G^\theta]_{\varpi_i}$ as a k-vector space.
- $Z_{\mathrm{Env}(G/G^\theta)} = A/(A \cap G^\theta)$, $A_{\mathrm{Env}(G/G^\theta)} = A/(A \cap G_\theta)$.
- $\mathbf{A}_{\mathrm{Env}(G/G^\theta)} = \mathrm{Spec}(k[e^{-\bar{\alpha}_i} : i = 1, \dots, l])$.

The multiplicative Hitchin map of a very flat symmetric embedding

Invariant theory for the symmetric embedding

(G, θ) as before. Σ very flat symmetric embedding such that the semisimple part of O_Σ is G/G^θ.

$$\Sigma /\!\!/ G^\theta = ((G/G^\theta) /\!\!/ G^\theta) \times \mathbf{A}_\Sigma.$$

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The multiplicative Hitchin map associated to Σ

Let X as before. We obtain a Hitchin-type fibration

$$\mathcal{M}_X(\Sigma) \xrightarrow{h_\Sigma} \mathcal{B}_X(\Sigma) \longrightarrow \mathcal{A}_X(\Sigma) \longrightarrow \mathrm{Bun}_{Z_\Sigma}(X)$$

by applying the functor $\mathrm{Map}(X, -)$ to the natural sequence of stacky quotients

$$[\Sigma/(G^\theta \times Z_\Sigma)] \longrightarrow [(\Sigma /\!\!/ G^\theta)/Z_\Sigma] \longrightarrow [\mathbf{A}_\Sigma/Z_\Sigma] \longrightarrow \mathbb{B}Z_\Sigma.$$

Comparing the two pictures

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (\mathsf{X}_*(A/(A \cap G^\theta))_-)^n \subset (\mathsf{X}_*(A/(A \cap G_\theta))_-)^n$. This defines

$$\begin{aligned} \boldsymbol{\lambda} : \mathbb{G}_m^n &\longrightarrow A/(A \cap G_\theta) \\ (z_1, \dots, z_n) &\longmapsto z_1^{\lambda_1} \cdots z_n^{\lambda_n}, \end{aligned}$$

which extends to a map $\boldsymbol{\lambda} : \mathbb{A}^n \to \mathbf{A}_{\mathsf{Env}(G/G^\theta)}$.

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which extends to a map $\boldsymbol{\lambda} : \mathbb{A}^n \to \mathbf{A}_{\mathsf{Env}(G/G^\theta)}$.
Consider $\Sigma_{\boldsymbol{\lambda}} = \mathsf{Env}(G/G^\theta) \times_{\boldsymbol{\lambda}} \mathbb{A}^n$ the corresponding very flat symmetric embedding. Note that $\mathbf{A}_{\Sigma_{\boldsymbol{\lambda}}} = \mathbb{A}^n$, so $\mathbb{B}Z_{\Sigma_{\boldsymbol{\lambda}}} = \mathsf{Pic}(X)^n$, and for any tuple of line bundles $\boldsymbol{L} = (L_1, \dots, L_n)$,

$$\mathcal{A}_X(\Sigma_{\boldsymbol{\lambda}})_{\boldsymbol{L}} = \bigoplus_{i=1}^{n} H^0(X, L_i).$$

Comparing the two pictures

Let $\boldsymbol{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ and $\boldsymbol{D} = (D_1, \ldots, D_n) \in X_{\boldsymbol{d}}$.
Since each D_i is effective, there exists a canonical section s_i of each $\mathcal{O}_X(D_i)$. Let us denote $\mathcal{O}_X(\boldsymbol{D}) = \bigoplus_{i=1}^n \mathcal{O}_X(D_i)$ and $\boldsymbol{s} = (s_1, \ldots, s_n)$.

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Theorem (G–García-Prada)

The map $X_{\boldsymbol{d}} \to \mathcal{A}_X(\Sigma_{\boldsymbol{\lambda}})$, $\boldsymbol{D} \mapsto (\mathcal{O}_X(\boldsymbol{D}), \boldsymbol{s})$ induces the following diagram, where all squares are Cartesian

$$\begin{array}{ccccc} \mathcal{M}_{\boldsymbol{d},\boldsymbol{\lambda}}(G,\theta) & \xrightarrow{h_{\boldsymbol{d},\boldsymbol{\lambda}}} & \mathcal{B}_{\boldsymbol{d},\boldsymbol{\lambda}}(G,\theta) & \longrightarrow & X_{\boldsymbol{d}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_X(\Sigma_{\boldsymbol{\lambda}}) & \xrightarrow{h_{\Sigma_{\boldsymbol{\lambda}}}} & \mathcal{B}_X(\Sigma_{\boldsymbol{\lambda}}) & \longrightarrow & \mathcal{A}_X(\Sigma_{\boldsymbol{\lambda}}). \end{array}$$

Contents

Involutions on multiplicative Higgs bundles

- Consider the natural map $\pi : \mathrm{Aut}_2(G) \to \mathrm{Out}_2(G) = \mathrm{Aut}_2(G)/\mathrm{Int}(G)$.

The involutions

Given any $a \in \mathrm{Out}_2(G)$ and $\varepsilon = \pm 1$, we can consider the involution

$$\iota_a^\varepsilon : (E, \varphi) \longmapsto (\theta(E), \theta(\varphi)^\varepsilon),$$

for any $\theta \in \pi^{-1}(a)$.

- Goal: Study the fixed points $((E, \varphi) \cong (\theta(E), \theta(\varphi)^\varepsilon))$.

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- Clearly, for any $\theta \in \pi^{-1}(a)$, multiplicative G^θ-Higgs bundles are fixed under ι_a^+ and multiplicative (G, θ)-Higgs bundles are fixed under ι_a^-.

A little bit more on involutions

- (G, θ) as before.
- G acts on itself by θ-twisted conjugation

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, s) &\longmapsto g * s = gs\theta(g)^{-1}. \end{aligned}$$

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- The orbits are homogeneous spaces of the form $G * s \cong G/G^{\theta_s}$, for $\theta_s = \mathrm{Int}_s \circ \theta$.
- θ_s is an involution if and only if $s \in S_\theta = \{s \in G : s\theta(s) \in Z_G\}$.
- $*$ can be extended to an action of $G \times Z_G$, which preserves S_θ.

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- $*$ can be extended to an action of $G \times Z_G$, which preserves S_θ.
- G/G^θ and $G/G^{\theta'}$ can be identified if θ and θ' are related by

$$\theta \sim \theta' \text{ if and only if there exists } \alpha \in \mathrm{Int}(G) \text{ such that } \theta' = \alpha \circ \theta \circ \alpha^{-1}$$

- $\pi : \mathrm{Aut}_2(G) \to \mathrm{Out}_2(G)$ descends to the clique map $\mathrm{cl} : \mathrm{Aut}_2(G)/\sim \to \mathrm{Out}_2(G)$ and

$$\mathrm{cl}^{-1}(a) \cong S_\theta/(G \times Z_G) = H^1_\theta(\mathbb{Z}/2, G^{\mathrm{ad}}),$$

for any $\theta \in \pi^{-1}(a)$.

The fixed points

Theorem (G–García-Prada)

Let $a \in \mathrm{Out}_2(G)$. *If* (E, φ) *is simple and* $(E, \varphi) \cong \iota_a^\varepsilon$, *then:*

1. *There exists a unique* $[\theta] \in \mathrm{cl}^{-1}(a)$ *such that there is a reduction of structure group of E to a* G^θ*-bundle* $E_\theta \subset E$.
2. *If we consider the corresponding G-equivariant map* $f_\varphi : E|_{X \setminus |D|} \to G$, *then* $f_\varphi|_{E_\theta}$ *takes values into* G^θ *if* $\varepsilon = 1$, *and in* $S^\theta := \left\{ s \in G : s = \theta(s)^{-1} \right\}$ *if* $\varepsilon = -1$.

More precisely, when $\varepsilon = -1$, $f_\varphi|_{E_\theta}$ *takes values in a single orbit* $G * s \subset S^\theta$ *for some* $s \in S^\theta$ *unique up to* θ*-twisted conjugation.*

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Proof idea

Pick $\theta_0 \in \pi^{-1}(a)$ and $\psi : (E, \varphi) \to (E, \varphi)$ a θ_0-twisted automorphism. Then we get $f_\psi : E \to S_{\theta_0}$ G-equivariant, so it maps to a single G-orbit G/G^θ. This gives the reduction to G^θ.

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By assumption we have $f_\psi(e)\theta_0(f_\varphi(e))f_\psi(e)^{-1} = f_\varphi(e)^\varepsilon$ so, for $e \in E_\theta$, we get $\theta(f_\varphi(e)) = f_\varphi(e)^\varepsilon$.

The symplectic structure

- (G, θ) as above.
- Assume that X is Calabi–Yau (so $X = \mathbb{A}^1$, $\mathbb{G}_m$ or an elliptic curve).
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Theorem (G–García-Prada)

For any $a \in \mathrm{Out}_2(G)$,

$$(\iota_a^{\varepsilon})^*\Omega = \varepsilon\Omega.$$

Therefore, the fixed points of ι_a^+ form an algebraic symplectic submanifold and the fixed points of ι_a^- form an algebraic Lagrangian submanifold.

Contents

The Charbonneau–Hurtubise correspondence

- Assume $k = \mathbb{C}$ and work in the analytic category.
- Multiplicative G-Higgs bundles on X are equivalent to mini-holomorphic G-bundles on $X \times S^1$ with Dirac-type singularities.
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- Let $\mathbb{E} \to X \times S^1$ be such a mini-holomorphic bundle.
- Given a reduction of structure group h from $\mathbb{E}$ to a maximal compact subgroup $K \subset G$ (a "Hermitian metric"), there is an associated Chern pair (A_h, ϕ_h) formed by a K-connection A_h and a "Higgs field" ϕ_h.
- If ad-hoc stability conditions are satisfied, one can use Donaldson–Uhlenbeck–Yau on $X \times S^1 \times S^1$ to obtain a reduction h such that the corresponding pair (A_h, ϕ_h) satisfies the Hermitian–Einstein–Bogomolny equation

$$F_{A_h} - iC\omega_X = *d_{A_h}\phi_h.$$

Involutions of mini-holomorphic bundles

- If (E, φ) is obtained by taking scattering of some mini-holomorphic bundle $\mathbb{E}$, then (E, φ^{-1}) is obtained by scattering in the opposite direction, thus it corresponds to $\epsilon^*\mathbb{E}$, for

$$\begin{aligned} \epsilon : S^1 &\longrightarrow S^1 \\ e^{it} &\longmapsto e^{i(2\pi - t)}. \end{aligned}$$

- Thus, in terms of mini-holomorphic bundles, we get the involutions

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- The description for monopoles should follow from here (future work with García-Prada).

Contents

Further directions

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- Applications to relative Langlands?

Thank you