Multiplicative Hitchin fibrations and Langlands duality

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Guillermo Gallego
(UAM—ICMAT)

C smooth complex projective curve

G semisimple complex group

$G^{sc} \longrightarrow G$ simply connected cover

Definition A <u>multiplicative G-Higgs bundle</u> over C is a pair (E, φ) :

- $E \to C$ principal G-bundle
- $\varphi \in \Gamma(C \setminus \{p_1, \ldots, p_n\}, E \times^{Ad(G)} G^{sc})$

$p_i \in C$

$$\left[\text{Fix:}\quad \begin{array}{ccccc} T^{sc} & \hookrightarrow & B^{sc} & \hookrightarrow & G^{sc} \\ \downarrow & & & & \downarrow \\ T & \hookrightarrow & B & \hookrightarrow & G \end{array}\right]$$

z local coord. near p_i,

Local type at p_i $\rightarrow$ $\varphi|_{\mathbb{D}_{p_i}} \in G[[z]] \backslash G^{sc}((z)) / G[[z]] \cong X_*(T^{sc})/W \cong X_+(T^{sc})$

STACK OF MULTIPLICATIVE G-HIGGS BUNDLES OF LOCAL TYPE D

$$\boxed{\begin{array}{c} \mathcal{M}_{G,D} = \langle (E,\varphi) \text{ mult. } G\text{-Higgs bundle on } C \text{ w. local type } D \rangle \\ D = \sum_{i=1}^{n} \lambda_i p_i, \qquad \lambda_i \in X_+(T^{sc}). \end{array}}$$

$(r = \mathrm{rk}\, G)$

- $\omega_1, \dots, \omega_r$ fundamental weights of G^{sc}

$\downarrow$

$\rho_1, \dots, \rho_r$ fundamental representations of G^{sc}

$g \in G^{sc}$

$p_i(g) := \mathrm{tr}\left(\rho_{\omega_i}(g)\right)$

MULTIPLICATIVE HITCHIN FIBRATION (Hurtubise – Markman)

$$h_{G,D} : \mathcal{M}_{G,D} \longrightarrow \mathcal{A}_{G,D} := \bigoplus_{i=1}^{r} H^0\big(C, \mathcal{O}_C(\underbrace{\langle D, \omega_i \rangle}_{= \sum_j \langle \lambda_j, \omega_i \rangle p_j})\big)$$

$$(E, \varphi) \longmapsto \left(p_1(\varphi), \dots, p_r(\varphi)\right)$$

Remarks

$\boxed{\text{simply connected!!}}$

1. "Global" version of $\left[G^{sc}/G\right] \longrightarrow G^{sc}/\!\!/G \cong T^{sc}/W \cong \mathbb{A}^r$

2. Contrast with usual Hitchin fibration: $\left[\mathfrak{g}/G\right] \longrightarrow \left[\mathfrak{g}/\!\!/G\right]$.

Remarks

3. There is a Hitchin–Kobayashi correspondence:

(Charbonneau–Hurtubise, Smith)

$$\underbrace{\mathcal{M}^{ps}_{G,D}}_{\downarrow \text{ local types}} \cong \left\{ \underbrace{\text{singular}}_{\downarrow \text{ Dirac-type singularities}} \ K\text{-monopoles on } \mathbb{C}\times S^1 \right\} \big/ \text{gauge}$$

local types $\longleftrightarrow$ Dirac-type singularities

$K \subset G$ maximal compact

[solutions to $F_\nabla = * \nabla \Phi$ Bogomolny eqn.]

Remarks

4. When $C = \underbrace{\mathbb{C}, \mathbb{C}^*}_{(\mathbb{CP}^1 \text{ w. framings})}$ or elliptic curve we have

$$\boxed{C \times S^1 \cong \mathbb{R}^3/\mathbb{Z}^{1\ 2\ 3}}$$

$\mathcal{M}_{G,D}$ hyperkähler and Mochizuki's NAHT:

$$\mathcal{M}_{G,D} \cong \begin{cases} \text{difference modules} \\ q\text{-difference modules} \\ \text{elliptic difference modules} \end{cases}$$

THE MONOID POV

Idea:

Want : $G^D \longrightarrow C$, partial compactification of G^{sc} over C,

$$\mathcal{M}_{G,D} = \mathrm{Map}_C\left(C, \left[G^D/G\right]\right) \qquad \mathcal{A}_{G,D} = \mathrm{Map}_C\left(C, G^D/\!\!/G\right).$$

[similar to $\mathrm{Higgs}_G = \mathrm{Map}_C\left(C, \left[\mathfrak{g}^{K_C}/G\right]\right)$, $\mathrm{Hitch}_G = \mathrm{Map}_C\left(C, \mathfrak{g}^{K_C}/\!\!/G\right)$.]

Solution: The Vinberg monoid (Frenkel-Ngô, Bouthier, Chi, Wang)

$$
\begin{tikzcd}
\dfrac{G^{sc} \times T^{sc}}{Z_{G^{sc}}} \arrow[r, hook] \arrow[d, "\alpha"'] & \mathrm{Vin}(G^{sc}) \arrow[d] \\
T^{ad} \arrow[r, hook] & \mathbb{A}^r
\end{tikzcd}
$$

$G \times G \circlearrowright \mathrm{Vin}(G^{sc}) \circlearrowleft T$, $\quad T \circlearrowright T^{ad}$, $\quad T \circlearrowright \mathbb{A}^r$

flat family of copies of G^{sc} degenerating at hyperplanes $\{\alpha_i = 0\}$

$\left[\lim_{z \to 0} z^{\lambda} \in \mathbb{A}^r \Leftrightarrow \lambda \in X_+(T^{sc})\right]$

$$
\begin{tikzcd}
G^D \arrow[r] \arrow[d] & {[\mathrm{Vin}(G^{sc})/T]} \arrow[d] \\
C \arrow[r, "{(\varphi_c(D),\, 1_D)}"'] & {[\mathbb{A}^r/T]}
\end{tikzcd}
$$

Example : $\mathrm{Vin}(SL_2) \cong \mathrm{Mat}_{2\times 2} \xrightarrow{\det} \mathbb{A}^1$

$$\begin{tikzcd}
SL_2^D \arrow[r] \arrow[d] & {[\mathrm{Mat}_{2\times 2}/\mathbb{G}_m]} \arrow[d, "\det"] \\
C \arrow[r, "{(\mathcal{O}_C(D),\, 1)}"'] & {[\mathbb{A}^1/\mathbb{G}_m]}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{M}_{SL_2,D} = \left\{ (E,\varphi) \;\middle|\; \begin{array}{ll} E \to C \text{ rk. 2 v.b.} & \det E \cong \mathcal{O}_C \\ \varphi : E \to E \otimes \mathcal{O}_C(D) & \det \varphi = 1 \end{array} \right\} \arrow[d, "\mathrm{tr}"] \\
\mathcal{A}_{SL_2,D} = H^0(C, \mathcal{O}_C(D))
\end{tikzcd}$$

SYMMETRIES AND CAMERAL DESCRIPTION

$$I_{G,D,x} = \{ g \in G \mid g x g^{-1} = x \}$$

$$\begin{tikzcd} I_{G,D} \arrow[r] \arrow[d, squiggly] & G^D \arrow[d] \\ \boxed{J_{G,D}} \arrow[r] & G^D /\!/ G \end{tikzcd}$$

REGULAR CENTRALIZER

$$\begin{tikzcd} J_{G,D,a} \arrow[r] \arrow[d] & J_{G,D} \arrow[d] \\ C \arrow[r, "a"'] & G^D /\!/ G \end{tikzcd}$$

$a \in \mathcal{A}_{G,D}$

PICARD STACK ACTING ON FIBRES

$$\boxed{\mathcal{P}_{G,D,a} = \mathrm{Bun}_{J_{G,D,a}/C}} \circlearrowright h_{G,D}^{-1}(a)$$

PICARD STACK ACTING ON FIBRES

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Has a "Galois" description in terms of the

CAMERAL COVER

$$\boxed{\begin{array}{ccc} \tilde{C}_a & \longrightarrow & T^D \subseteq G^D \\ \downarrow & & \downarrow \\ C & \xrightarrow{a} & T^D/W \cong G^D /\!/ G \end{array}}$$

Thm [Hurtubise - Markman, Bouthier, Chi, Wang]

Assume:

- D "ample" $\longrightarrow$ condition on $\deg\left(\sum_i \langle D, w_i \rangle\right)$.
- a $\underbrace{\text{"generic"}}$ $\longrightarrow$ cut boundary or discriminant but not both

group-like locus / HM-regular

Then:

- $\tilde{C}_a$ is smooth
- $P_{G,D,a}$ is a Beilinson 1-Motive $\leftarrow$ more on these now
- $h^{-1}(a)$ is a $P_{G,D,a}$-torsor (trivializable)
- $\tilde{C}_a \to C$ cameral cover of type $\mathfrak{g}$ and $J_{G,D,a} \cong J_G$

[BEILINSON 1-MOTIVES]

(fpqc-locally) abelian variety

$$\mathcal{P} \cong \Gamma \times P \times BH$$

f.g. abelian group

group of multiplicative type

duality for Picard stacks

dual ab. variety

$$\mathrm{Dual}(\mathcal{P}) = \mathrm{Hom}(\mathcal{P}, B\mathbb{G}_m) = B\Gamma^* \times \hat{P} \times H^*$$

[CARTIER DUALITY]

$$\left[\begin{array}{l} H^* = \mathrm{Hom}(H, \mathbb{G}_m) \\ \Gamma^* = \mathrm{Hom}(\Gamma, \mathbb{G}_m) \end{array}\right]$$

$$\mathrm{FM}: \quad \mathrm{Coh}(\mathcal{P}) \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Dual}(\mathcal{P}))$$

$$\mathrm{FM}: \mathrm{Coh}(P) \to \mathrm{Coh}(\hat{P}) \quad \leftarrow \text{Mukai} \qquad \text{(derived)}$$

$$\mathrm{Rep}(H) \longleftrightarrow H^*$$

REVIEW OF DONAGI-PANTEV DUALITY

$\mathfrak{g}$ complex semisimple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ Cartan

W Weyl group of $\mathfrak{g}$

$\tilde{C} \xrightarrow{\pi} C$ ramified Galois W-cover of smooth curves

Def: This
- is a camenal curve of type $\mathfrak{g}$ if it is étale-locally iso. to a pullback of $\mathfrak{t} \longrightarrow \mathfrak{t}/W$ ← standard cameral cover
- has simple Galois ramification if $\mathrm{im}(C \to \mathfrak{t}/W)$ intersects transversely the discriminant divisor $\mathcal{D} \subset \mathfrak{t}/W$.

← 0's of roots

G complex semisimple group w. $\mathrm{Lie}(G) = \mathfrak{g}$ | $G^\vee$ Langlands dual

$J_G \to C$ group scheme defined as $J_G \underset{\substack{\text{open}\\\text{closed}}}{\hookrightarrow} \pi_*(\tilde{C} \times T)^W$

[Donagi - Gaitsgory]

[on ramification points x

$\mathrm{Ker}\, \alpha_x = J_{G,x} \hookrightarrow T^{S_{\alpha_x}}$]

The regular centralizers of usual Hitchin fibrations are like this.

$$\begin{tikzcd} \mathcal{P}_G = \mathrm{Bun}_{J_G/C} \arrow[dr] & \circlearrowright & \mathrm{Higgs}_G \arrow[dl, shift left] \\ & \mathrm{Hitch}_G \arrow[ur, shift left] & \end{tikzcd}$$

Hitchin section trivializes

$$\begin{tikzcd} \mathfrak{t} \arrow[rr, "\text{Killing form}", "\sim"'] \arrow[dr] & & \mathfrak{t}^* \arrow[dl] \\ & \mathfrak{t}/W \cong \mathfrak{t}^*/W & \end{tikzcd}$$

$\rightsquigarrow$

$$\boxed{\text{Hitchin bases and cameral covers for dual groups are identified}}$$

Thm [Hausel-Thaddeus [type A], Donagi-Pantev [in general], Chen-Zhu [char. ≠ 0]] ← [already hinted by Hitchin]

$a \in \mathrm{Hitch}_G^\diamond$ ← [locus w. smooth cameral covers]

$\cong$ [via the Killing form ($\mathfrak{t} \cong \mathfrak{t}^*$)]

$\mathrm{Hitch}_{G^\vee}^\diamond$

$$\boxed{P_{G,a} = \mathrm{Dual}\left(P_{G^\vee,a}\right)}$$

Topological argument: [$Z_G^* \cong \pi_1(G^\vee)$]

$$H_1(P_{G,a}) \cong H_1(P_{G^\vee,a})^\vee$$

[For char ≠ 0, using Tate modules by Chen and Zhu]

Cor.

$$\mathrm{Coh}\left(\mathrm{Higgs}_G / \mathrm{Hitch}_G^\diamond\right) \xrightarrow[\sim]{FM} \mathrm{Coh}\left(\mathrm{Higgs}_{G^\vee} / \mathrm{Hitch}_G^\diamond\right)$$

BONUS compatible w. "abelianized" Hecke - Wilson
(t' Hooft)

[Hitchin section]

MULTIPLICATIVE DUALITY

roots of equal length ($\rightsquigarrow$ normalize it)

· In the <u>simply laced</u> case:

$$\alpha \longmapsto (\alpha, -) = \alpha^\vee$$

$$X_*((T^\vee)^{sc}) = X^*(T^{ad}) \cong \Lambda_{root}(\mathfrak{g}) \xrightarrow{\sim} \Lambda_{coroot}(\mathfrak{g}) \cong X_*(T^{sc})$$

<u>Killing</u>:

$$\begin{tikzcd}
\Lambda_{root}(\mathfrak{g}) \arrow[r, "\sim"] \arrow[d, phantom, "\cap" description] & \Lambda_{coroot}(\mathfrak{g}) \arrow[d, phantom, "\cap" description] \\
\mathfrak{t}^* \arrow[r, "W\text{-equiv.}", "\sim"'] \arrow[d] & \mathfrak{t} \arrow[d] \\
(T^\vee)^{sc} \arrow[r, "W\text{-eq.}", "\sim"'] & T^{sc}
\end{tikzcd}$$

$\rightsquigarrow$ $\boxed{\mathcal{A}_{G^\vee, D} \cong \mathcal{A}_{G, D}}$

[Conjectured by Elliott-Pestun] → [Duality of 5D theories twisted on S^1]

Thm [G.] If G is simply laced

$$\boxed{\mathcal{P}_{G,a} = \mathrm{Dual}\left(\mathcal{P}_{G^\vee,a}\right)} .$$

$a \in \mathcal{A}_{G,D} \cong \mathcal{A}_{G^\vee,D}$

$\mathcal{M}_{G,D,a}$ $\mathcal{M}_{G^\vee,D,a}$

Proof. $\tilde{C}_a \to C$ cover of type $\mathfrak{g}$.

$J_{G,a} \cong J_G \rightsquigarrow \mathcal{P}_{G,a} = \mathrm{Bun}_{J_G/C}$

$J_{G^\vee,a} \cong J_{G^\vee} \rightsquigarrow \mathcal{P}_{G^\vee,a} = \mathrm{Bun}_{J_{G^\vee}/C}$

Dual by Donagi-Pantev

Example

$$\mathcal{M}_{SL_2,D} = \left\langle (E,\varphi) \;\middle|\; \begin{array}{l} E \to C \text{ rk. } 2,\ \det E \xrightarrow{\sim} \mathcal{O}_C \\ \varphi : E \to E \otimes \mathcal{O}(D),\ \det\varphi = 1 \end{array} \right\rangle$$

$$\Big\downarrow\ tr = h_D$$

$$\mathcal{A} = H^0(C, \mathcal{O}(2D))$$

$$\ni a$$

SPECTRAL CURVE $\to S_a = \{ y^2 + ay + 1 = 0 \}$

$$\downarrow$$

$$C$$

$$\mathcal{P}_{SL_2,a} \cong \mathrm{Prym}(S_a/C) \overset{\boxed{\text{BNR}}}{\circlearrowright} h_D^{-1}(a)$$

$$\mathcal{M}_{PGL_2,D} = \left[\mathcal{M}_{SL_2,D} \big/ \mathrm{Jac}(C)[2] \right] \rightsquigarrow \mathcal{P}_{PGL_2,a} = \mathrm{Prym}(S_a) \big/ \mathrm{Jac}(C)[2] = \mathrm{Prym}(S_a)^\vee$$

- If G is <u>not simply-laced</u>:

  - Killing

$$\begin{tikzcd} W \circlearrowright \mathfrak{t}^* \arrow[r, "\sim"] & \mathfrak{t} \circlearrowleft W \\ \Lambda_{\mathrm{root}}(\mathfrak{g}) \arrow[u, hook] \arrow[r, "\times" marking] & \Lambda_{\mathrm{coroot}}(\mathfrak{g}) \arrow[u, hook] \end{tikzcd}$$

✓

  - but $\Lambda_{\mathrm{root}}(\mathfrak{g}) \not\to \Lambda_{\mathrm{coroot}}(\mathfrak{g})$.

<u>Problem</u>: [We cannot match T^{sc}/W and $(T^\vee)^{sc}/W$ in this case.]

<u>Solution</u>: [Twisted multiplicative Hitchin fibrations]

If $H = G^\theta$ with $\left[\begin{array}{l} G \text{ simply laced simply connected} \\ \theta \text{ diagram automorphism} \end{array}\right]$

↳ classified by affine Dynkin diagrams

Then,

$$\boxed{G\theta /\!\!/ G \cong (H^\vee)^{sc} /\!\!/ H^\vee} \cong \frac{T/(1-\theta)(T)}{W^\theta}$$

[Mohrdieck]

Def. Let (G,θ) as above.

A **θ-twisted multiplicative G-Higgs bundle** on C is (E,φ)

- $E \to C$ principal G-bundle
- $\varphi \in \Gamma\left(C \setminus \{p_1,\dots,p_m\},\ E \times^{\boxed{G,\theta}} G\right)$

twisted adjoint action
$g * x = g x \theta(g)^{-1}$

[same local types as untwisted G-Higgs]

[Griffin Wang, unpublished] $\rightsquigarrow$ Study of regular centralizers for $G \underset{\theta}{\circlearrowleft} G$

[Twisted endoscopy]

[Excluding $A_{2\ell}^{(2)}$]

↳ cameral covers of type h

↳ $J_{(G,\theta)} \cong J_H$ $\left[\cong \pi_*\left(\tilde{C}_a \times T_H\right)^{W^\theta}\right]$

["Conjectured" by Elliott–Pestun] → [Duality of 5D theories twisted on S^1]

Thm [G.] For (G,θ) as above, $H = G^\theta$,

[G^θ is simply connected]

$(H^\vee)^{sc} = G_\theta$

$T \longrightarrow T_\theta$

$$\boxed{\mathcal{P}_{(G,\theta),a} = \mathrm{Dual}\left(\mathcal{P}_{H^\vee,a}\right)}$$

$\mathcal{M}_{(G,\theta),D,a}$ $\qquad$ $\mathcal{M}_{H^\vee,D,a}$

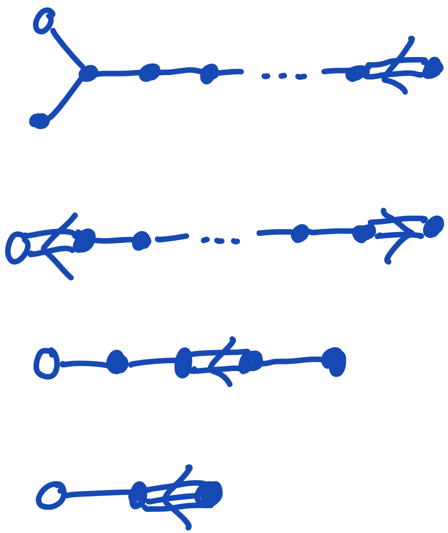
Examples

$$(SL_{2\ell}, SL_{2\ell}\theta) \leftrightarrow (SO_{2\ell+1}, Spin_{2\ell+1})$$

$$(Spin_{2\ell+2}, Spin_{2\ell+2}\theta) \leftrightarrow (PSp_{2\ell}, Sp_{2\ell})$$

$$(E_6, E_6\theta) \leftrightarrow (F_4, F_4)$$

$$(Spin_8, Spin_8\theta_3) \leftrightarrow (G_2, G_2)$$



$$A_{2\ell-1}^{(2)} = \left(B_\ell^{(1)}\right)^\vee$$

$$D_{\ell+1}^{(2)} = \left(C_\ell^{(1)}\right)^\vee$$

$$E_6^{(2)} = \left(F_4^{(1)}\right)^\vee$$

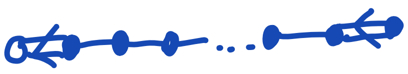
$$D_4^{(3)} = \left(G_2^{(1)}\right)^\vee$$

Special cases

$$(SL_3, SL_3\theta)$$

$$A_2^{(2)}$$

$$(SL_{2\ell+1}, SL_{2\ell+1}\theta)$$



$$A_{2\ell}^{(2)}$$

[Physics: Discrete theta-angle] ← [e.g. Tachikawa ...]

$A_2^{(2)}$ explicitly

$G = SL_3, \quad \theta \sim g \mapsto (g^T)^{-1}.$

Moduli stack: $\displaystyle \mathcal{M} = \Big\{ (V, \ , \nu) \ \Big| \ V \to C \ \text{rk. 3 v.b.}, \ \gamma : V \to V^* \otimes \underbrace{L}_{\text{fixed line bundle}}, \ \nu : \det V \xrightarrow{\sim} \mathcal{O}_C \Big\}$

"Hitchin map":

$$\boxed{\begin{array}{rcl} h : \mathcal{M} & \longrightarrow & H^0(C, L^3) \oplus H^0(C, L^3) \\ (V, \gamma, \nu) & \longmapsto & \Big(\underbrace{s(\gamma)}_{\text{[the "meromorphic datum"]}} , \quad t(\gamma) \Big) \end{array}}$$

$\gamma = \underset{\text{symmetric}}{g} + \underset{\text{skew-sym.}}{\omega}, \qquad \omega \in H^0(\overbrace{\Lambda^2 V^*}^{\cong V} \otimes L) \cong \mathrm{Hom}(L^{-1}, V) \ni \sigma_\omega \qquad [\omega = \sigma_\omega \times -]$

$s(\gamma) = \det(g), \quad t(\gamma) = g(\sigma_\omega, \sigma_\omega) \qquad \Big[\det(x_0 \gamma + x_1 \gamma^T) = (x_0 + x_1)\big[s(x_0+x_1)^2 + t(x_0 - x_1)^2\big]\Big]$

Spectral data $s,t \in H^0(C,L^3)$ $\quad \sigma_{s,t}(p,(x_0:x_1)) = s(p)x_0^2 + t(p)y_1^2$

$$\begin{tikzcd} S_{s,t} = (\sigma_{s,t})_0 \arrow[r, hook] \arrow[dr, "\pi"'] & C \times \mathbb{P}^1 \arrow[d] \\ & C \end{tikzcd}$$

spectral curve ↑

Thm [G.]

Sup. $S_{s,t}$ smooth.

$$h^{-1}(s,t) = \left\langle (U,\eta,\psi) \;\middle|\; \begin{array}{l} U \to S_{s,t} \text{ line bundle} \\ \eta : L^4 \xrightarrow{\sim} \mathrm{Nm}_\pi U \\ \psi = (\psi_p)_{p\in(t)_0},\ \psi_p \in (\pi_* U)_p^* \otimes L_p^2 \end{array} \quad + (*) \right\rangle$$

compatibility condition

Corol.

$$h^{-1}(s,t) \circlearrowleft P_{s,t} \leftarrow \text{quotient of } \mathrm{Prym}(S_{s,t}),$$

$$\boxed{\hat{P}_{s,t} = P_{t,s}}$$

[Can be seen as self-duality of mult. Hitch. w. automorphism on the base.]

References

- G., Multiplicative Hitchin fibrations and Langlands duality. 2025 [arXiv: 2509.14364] (submitted)
- C. Elliott and V. Pestun, Multiplicative Hitchin systems and susy gauge theory. 2019 [Selecta Math.]
- X. Griffin Wang. Multiplicative Hitchin fibrations and Fundamental Lemma. 2024 [to appear in Ann. of Math. Studies]
- X. Griffin Wang. Notes on twisted endoscopy. [unpublished]

Thanks for your attention