UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

MULTIPLICATIVE HIGGS BUNDLES, MONOPOLES AND INVOLUTIONS

Fibrados de Higgs multiplicativos, monopolos e involuciones

MEMORIA PARA OPTAR AL GRADO DE DOCTOR PRESENTADA POR

GUILLERMO GALLEGO SÁNCHEZ

TRABAJO DIRIGIDO POR

ENRIQUE ARRONDO ESTEBAN

ÓSCAR GARCÍA PRADA

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DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD DE LA TESIS PRESENTADA PARA OBTENER EL TÍTULO DE DOCTOR

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de la Facultad de Ciencias Matemáticas de la Universidad Complutense de Madrid, como autor/a de la tesis presentada para la obtención del título de Doctor y titulada:

Fibrados de Higgs multiplicativos, monopolos e involuciones
Multiplicative Higgs bundles, monopoles and involutions

y dirigida por: Enrique Arrondo Esteban y Óscar García Prada

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A mis padres

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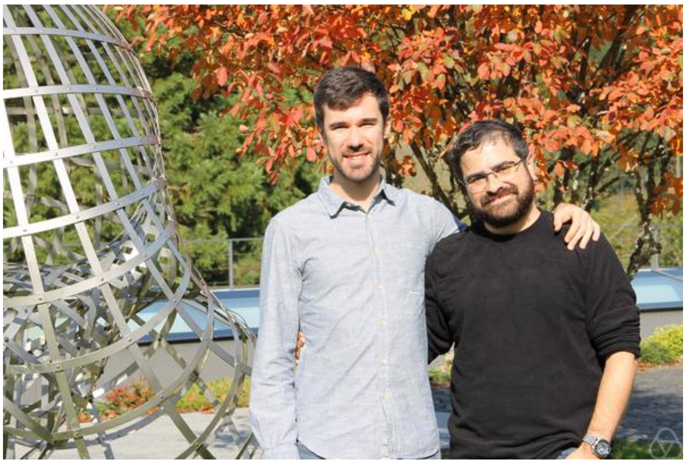

Figura 0.1: Guille B (izquierda) y yo. (Fuente: Archivos del Mathematisches Forschungsinstitut Oberwolfach, autora: Petra Lein).

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CONTENTS

ABSTRACT

This dissertation is centered around the study of the action of a holomorphic involution θ of a complex reductive group G on the space of multiplicative G-Higgs bundles over a compact Riemann surface X, as introduced by Hurtubise and Markman [HM02]. Provided the correspondence of Charbonneau–Hurtubise [CH11] and Smith [Smi16] between multiplicative Higgs bundles and singular monopoles on $S^1 \times X$, where S^1 is the circle, equivalently we study the action of θ on the moduli space of singular monopoles on $S^1 \times X$. More precisely, we study the fixed points of the involutions

$$\iota_{\pm}^{\theta} : (E, \varphi) \longmapsto (\theta(E), \theta(\varphi)^{\pm 1}).$$

Here, (E, φ) is a multiplicative G-Higgs bundle, $\theta(E)$ is the G-bundle associated to the action of θ on G, and $\theta(\varphi)$ is the natural section of $\theta(E)$ induced from φ.

In the process of describing the fixed points, we introduce a notion of multiplicative Higgs bundles with values on the symmetric variety G/G^{θ}, for $G^{\theta} \subset G$ the subgroup of fixed points of θ, which we call *multiplicative* (G, θ)-*Higgs bundles*. These objects appear as fixed points of the involution ι_{-}^{θ}, but other related objects also appear. The theory of multiplicative (G, θ)-Higgs bundles can be understood as a "multiplicative analog" of the theory of Higgs bundles for real groups. Higgs bundles for real groups already play an important role in the original paper of Hitchin introducing Higgs bundles [Hit87a] and in his later paper [Hit92], and their study has been very active in the last two decades; we refer the reader to [GP20] for a survey on this topic, and for further references therein. Taking into account that symmetric varieties are in fact a generalization of groups, the theory developed here provides a generalization of the theory of multiplicative Higgs bundles. Moreover, by replacing Vinberg's theory of reductive monoids [Vin95] by Guay's theory of embeddings of symmetric varieties [Gua01], we generalize some of the results of Bouthier, J. Chi and G. Wang [Bou15, BC18, Bou17, Chi22, Wan23] to the context of multiplicative (G, θ)-Higgs bundles.

When X has genus 1, we also study the interplay of the involutions $\iota_{\pm}^{\theta}$ with the holomorphic symplectic structure of the moduli space of multiplicative G-Higgs bundles and, equivalently, with the hyper-Kähler structure of the moduli space of singular monopoles on $S^1 \times X$. In particular, we show that the fixed points of ι_{+}^{θ} define a holomorphic symplectic submanifold of the moduli space, while the fixed points of ι_{-}^{θ} form a holomorphic Lagrangian submanifold.

Keywords: *Multiplicative Higgs bundle, multiplicative Hitchin fibration, involutions, symmetric varieties, mini-holomorphic bundle, monopoles, Bogomolny equations*

2020 MSC: Primary 14D23; Secondary 14H60, 14M17, 53C07

RESUMEN (EN ESPAÑOL)

Esta memoria se centra en el estudio de la acción de una involución holomorfa θ de un grupo reductivo complejo G en el espacio de los G-fibrados de Higgs multiplicativos sobre una superficie de Riemann compacta X, introducidos por Hurtubise y Markman [HM02]. Dada la correspondencia de Charbonneau–Hurtubise [CH11] y Smith [Smi16] entre fibrados de Higgs multiplicativos y monopolos singulares en $S^1 \times X$, donde S^1 es la circunferencia, equivalentemente estudiamos la acción de θ en el espacio de móduli de monopolos singulares en $S^1 \times X$. Concretamente, estudiamos los puntos fijos de las involuciones

$$\iota_{\pm}^{\theta} : (E, \varphi) \longmapsto (\theta(E), \theta(\varphi)^{\pm 1}).$$

Aquí, (E, φ) es un G-fibrado de Higgs multiplicativo, $\theta(E)$ es el G-fibrado associado a la acción de θ en G, y $\theta(\varphi)$ es la sección natural de $\theta(E)$ inducida por φ.

En el proceso de describir los puntos fijos, introducimos una noción de fibrados de Higgs multiplicativos con valores en la variedad simétrica G/G^{θ}, para $G^{\theta} \subset G$ el subgrupo de los puntos fijos de θ, que llamamos *(G, θ)-fibrados de Higgs multiplicativos*. Estos objetos aparecen como puntos fijos de la involución ι_{-}^{θ}, pero también aparecen otros objetos relacionados. La teoría de (G, θ)-fibrados de Higgs multiplicativos se puede entender como un «análogo multiplicativo» de la teoría de fibrados de Higgs para grupos reales. Los fibrados de Higgs para grupos reales ya juegan un papel importante en el artículo original de Hitchin [Hit87a] y en su artículo posterior [Hit92], y su estudio ha estado muy activo en los últimos veinte años; referimos a [GP20] para una reseña del tema y para más referencias. Entendiendo las variedades simétricas como una generalización de los grupos, la teoría desarrollada aquí da una generalización de la teoría de fibrados de Higgs multiplicativos. Reemplazando la teoría de monoides reductivos de Vinberg [Vin95] por la teoría de inmersiones de variedades simétricas de Guay [Gua01], generalizamos algunos de los resultados de Bouthier, J. Chi y G. Wang [Bou15, BC18, Bou17, Chi22, Wan23] a este contexto.

Cuando X tiene género 1, también estudiamos la interacción entre las involuciones $\iota_{\pm}^{\theta}$ con la estructura holomorfa simpléctica del espacio de móduli de G-fibrados de Higgs multiplicativos y, equivalentemente, con la estructura hiper-kähleriana del espacio de móduli de monopolos singulares en $S^1 \times X$. En particular, mostramos que los puntos fijos de ι_{+}^{θ} definen una subvariedad holomorfa simpléctica del espacio de móduli, mientras que los puntos fijos de ι_{-}^{θ} forman una subvariedad holomorfa lagrangiana.

Palabras clave: *Fibrado de Higgs multiplicativo, fibración de Hitchin multiplicativa, involuciones, variedades simétricas, fibrado mini-holomorfo, ecuaciones de Bogomolny*

2020 MSC: Primary 14D23; Secondary 14H60, 14M17, 53C07

Vino, primero, pura,
vestida de inocencia.
Y la amé como un niño.

Luego se fue vistiendo
de no sé qué ropajes.
Y la fui odiando, sin saberlo.

Llegó a ser una reina,
fastuosa de tesoros...
¡Qué iracundia de yel y sin sentido!

. . . Mas se fue desnudando.
Y yo le sonreía.

Se quedó con la túnica
de su inocencia antigua.
Creí de nuevo en ella.

Y se quitó la túnica,
y apareció desnuda toda. . .
¡Oh pasión de mi vida, poesía
desnuda, mía para siempre!

Juan Ramón Jiménez, *Eternidades*

INTRODUCTION

A PANORAMIC VIEW

A *multiplicative* G-*Higgs bundle* on X is a pair (E, φ), where E is a holomorphic principal G-bundle over X and φ is a meromorphic section of the adjoint group bundle $E(G) := E \times_G G$. This is indeed a "multiplicative" version of a (twisted) G-*Higgs bundle* on X, which is a pair (E, φ) with E a holomorphic principal G-bundle over X and φ a section of the adjoint Lie algebra bundle $E(\mathfrak{g})$, twisted by some line bundle L over X. In the multiplicative case, instead of fixing the twisting line bundle L, one controls the singularity at each singular point x by fixing a dominant cocharacter of G.

Charbonneau and Hurtubise [CH11] and later Smith [Smi16] have defined *stability* conditions for multiplicative G-Higgs bundles and proven a *Hitchin–Kobayashi type* correspondence between stable multiplicative G-Higgs bundles and *singular* K-*monopoles* over the 3-manifold $S^1 \times X$, where S^1 denotes the circle and $K \subset G$ is a maximal compact subgroup. Here, by a singular K-monopole over $S^1 \times X$ we mean a triple $(\mathbb{E}, \nabla, \Phi)$, with $\mathbb{E}$ a principal K-bundle over $S^1 \times X$, ∇ a K-connection on $\mathbb{E}$ and Φ a section of the adjoint bundle $\mathbb{E}(\mathfrak{k})$, such that

1. $(\mathbb{E}, \nabla, \Phi)$ can be locally approximated by a standard Dirac T_K-monopole over $\mathbb{R}^3$, for $T_K \subset K$ a maximal torus, and

2. it solves the *Hermitian–Bogomolny equation*

$$F_\nabla - *\nabla\Phi = iC\omega_X,$$

   for some central element $C \in Z(\mathfrak{k})$. Here F_∇ denotes the curvature of ∇, and $*$ is the Hodge star operator.

Consider now the involution $\theta \in \mathrm{Aut}_2(G)$. Using this involution, we can construct two different multiplicative G-Higgs bundles starting from a given one (E, φ), indeed, we can put

$$\iota_+^\theta(E, \varphi) = (\theta(E), \theta(\varphi)) \quad \text{and} \quad \iota_-^\theta(E, \varphi) = (\theta(E), \theta(\varphi)^{-1}).$$

Note that applying the same process again, we get back the starting (E, φ). The main purpose of this thesis is to describe the fixed points of these involutions. Of course, by going through the correspondence of Charbonneau–Hurtubise–Smith, one can consider the counterparts of these involutions in the moduli space of monopoles. In this document we describe precisely what this involution is in the moduli space of monopoles and give another equivalent description of the fixed points from the point of view of monopoles.

Among the different classes of objects appearing as fixed points of the involution ι_-^θ, there is one of particular interest, which is the one formed by what we call *multiplicative* (G,θ)*-Higgs bundles*. These provide a generalization of multiplicative G-Higgs bundles. A multiplicative (G,θ)-Higgs bundle on X is a pair (E,φ) with $E \to X$ a holomorphic principal G^θ-bundle and φ a meromorphic section of the associated bundle of symmetric varieties $E(G/G^\theta)$.

When G is semisimple, Frenkel and Ngô [FN11] suggest the idea that, if (E,φ) is a multiplicative G-Higgs bundle, then one can regard the section φ as a holomorphic (instead of meromorphic) map, if one extends it to a *very flat reductive monoid* such that the derived group of its unit group is equal to G. This idea has been crucial in the works of Bouthier and J. Chi dedicated to study analogues of affine Springer fibres [Bou15, Bou17, BC18, Chi22], and has ultimately led to the proof of the Fundamental Lemma of Langlands–Shelstad by G. Wang [Wan23], in the spirit of Ngô's work on the Fundamental Lemma for the Lie algebras [Ng10].

Here, we generalize the "monoid point of view" and some of the results of Bouthier, Chi and Wang to the context of multiplicative (G,θ)-Higgs bundles. In order to do this, we use Guay's theory of *very flat embeddings* of symmetric varieties [Gua01] and show that the section φ of a multiplicative (G,θ)-Higgs bundle can be extended to a very flat embedding Σ of the symmetric variety G/G^θ. Motivated by the results of Wang, we expect that our theory of multiplicative (G,θ)-Higgs bundles provides the correct framework for a proof of a generalization of the Fundamental Lemma for the groups to symmetric varieties, in the spirit of the Relative Langlands Program of Ben-Zvi–Sakellaridis–Venkatesh [BZSV23]. We leave that direction completely unexplored here.

An analog of the Hitchin fibration [Hit87b] exists in the context of multiplicative Higgs bundles, and was originally considered by Hurtubise–Markman [HM02]. The multiplicative Hitchin fibration has also been studied from the "monoid point of view" in the works of Frenkel–Ngô, Bouthier, Chi and Wang and can also be generalized to multiplicative (G,θ)-Higgs bundles. The study of the fibres and symmetries of Hitchin map for (G,θ)-Higgs bundles is not considered here, although we include some comments in the Further Directions section at the end of this dissertation.

Multiplicative (G,θ)-Higgs bundles are the multiplicative analog of Higgs bundles "for real groups", which under the *nonabelian Hodge correspondence* are matched to representations of the fundamental group of X in the real form of G associated to θ. A "multiplicative version" of the nonabelian Hodge correspondence is still not known, but expected at least in the case in which X has genus 1, since in that case the moduli space of monopoles is naturally hyper-Kähler. Our description of fixed monopoles under the involution ι_-^θ should explain what are the objects corresponding to them in this conjectural "de Rham side".

The Hitchin fibration is known to exhibit some form of *Langlands duality and mirror symmetry*. More precisely, the work of Donagi and Pantev [DP12] shows how the Hitchin bases associated to a reductive group G and its Langlands dual $\check{G}$ are naturally identified through the choice of an invariant bilinear form, while the Hitchin fibres are dual abelian varieties. Taking Fourier–Mukai transforms over these fibres is conjectured to give the construction of (B,B,B)-branes on the moduli space of $\check{G}$-Higgs bundles from (B, A, A)-branes on the moduli space of G-Higgs bundles. Higgs bundles for real forms $G_{\mathbb{R}}$ are known to define the support of a (B, A, A)-brane, conjecturally dual to the moduli space of $\check{G}_{G_{\mathbb{R}}}$-Higgs bundles, for $\check{G}_{G_{\mathbb{R}}}$ the Nadler dual group of the real form $G_{\mathbb{R}}$. In this document we show how multiplicative (G, θ)-Higgs bundles (and more generally, the fixed points of ι^{θ}_{-}) form the support of a (B, A, A)-brane on the moduli space of multiplicative G-Higgs bundles (when X has genus 1). We conjecture that results analogous to those of Donagi–Pantev hold in the multiplicative case, and that the dual (B, B, B)-brane corresponding to multiplicative (G, θ)-Higgs bundles is formed by multiplicative $\check{G}_{G_{\mathbb{R}}}$-bundles.

THE CLASSICAL STORY

Higgs bundles and the Hitchin moduli space

The theory of Higgs bundles was initiated by Hitchin [Hit87a] in the context of the study of the dimensional reduction of the self-dual Yang–Mills equations on $\mathbb{R}^4$ by the action of translation in two directions. The resulting equations in $\mathbb{R}^2$ are conformally invariant, and thus they can be considered over a compact Riemann surface X. This way, one obtains what are known as the *Hitchin equations*:

$$\begin{cases} F_{\nabla} + [\varphi, \varphi^{\dagger}] = -2\pi i \frac{d}{r} \, \mathrm{id}_{\mathbb{E}} \, \omega_X, \\ \nabla^{0,1} \varphi = 0. \end{cases}$$

These are equations for a pair (∇, φ), where ∇ is a Hermitian connection on a Hermitian vector bundle $\mathbb{E} \to X$ of rank r and degree d, and $\varphi \in \Omega^{1,0}(X, \operatorname{End} \mathbb{E})$ is a $(1, 0)$-form with values on the endomorphism bundle $\operatorname{End} \mathbb{E}$. The notation $[\varphi, \varphi^{\dagger}]$ indicates taking the commutator in the matrix part, and the wedge product in the form part. We use ω_X to denote the area form on X, that we choose to be normalized to total area 1. The operator ∇ endows $\mathbb{E}$ with the structure of a holomorphic vector bundle, which we denote by E, while the equation $\nabla^{0,1}\varphi = 0$ implies that we can regard φ as a holomorphic twisted endomorphism $\varphi : E \to E \otimes K_X$, for K_X the canonical line bundle of X. Such a pair (E, φ) is called a *Higgs bundle*.

One can easily check that Higgs bundles (E, φ) coming from an irreducible solution to the Hitchin equations must verify the *stability condition* which states that every φ-invariant subbundle $F \subset E$ must have a lower degree than E. Reciprocally, it is a theorem of Hitchin [Hit87a] (and of Simpson [Sim88], in broader generality) that every stable Higgs bundle arises in this way.

The *Hitchin moduli space* or *moduli space of Higgs bundles* can be constructed either with algebro-geometric tools, using Grothendieck's quot scheme and GIT, as in Nitsure [Nit91] and Simpson [Sim94a, Sim94b], or as a Kähler quotient, as in Hitchin's paper [Hit87a]. The equivalence of both constructions is a consequence of the theorem of Hitchin and Simpson.

Moreover, the Hitchin moduli space can in fact be obtained as a hyper-Kähler quotient and thus it is a hyper-Kähler manifold. We recall that a hyper-Kähler manifold is a Riemannian manifold endowed with three complex structures I, J and K which are Kähler with respect to the Riemannian metric and that satisfy the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$. When the Hitchin moduli space is constructed in this way, the structure I is clearly the one arising naturally from it parametrizing Higgs bundles. The other complex structures arise from a different moduli space, as we explain below.

The nonabelian Hodge correspondence

If (∇, φ) is a solution to the Hitchin equations, we can consider the operator

$$D = \nabla + \varphi - \varphi^{\dagger},$$

which defines a complex connection on the bundle $\mathbb{E}$. Moreover, since we assumed that the Hitchin equations hold, the connection D is flat. Taking the holonomy, we obtain an r-dimensional representation of the fundamental group of X. Reciprocally, it is a theorem of Corlette [Cor88] (and of Donaldson [Don87] in the original setting of Hitchin's paper [Hit87a]) that flat bundles which come from *reductive* representations admit a metric which is *harmonic*, which implies that the connection D comes from a solution to the Hitchin equations.

The moduli space classifying representations is known as the *character variety* and it is naturally an affine variety over $\mathbb{C}$. As a consequence of the results of Corlette and Donaldson, the character variety is also homeomorphic to the Hitchin moduli space (actually, these spaces are diffeomorphic in the smooth locus, and, in fact, they are real-analitically isomorphic). The (isomorphic) complex structures J and K can be understood as coming from the natural complex structure of the character variety.

The theory of Higgs bundles, the Hitchin equations, and all the results of Corlette, Donaldson, Hitchin and Simpson remain valid when one considers principal G-bundles, for G a complex reductive group, instead of vector bundles (and one would recover the vector bundle case for $G = GL_r(\mathbb{C})$).

The Hitchin fibration

Another very important feature of the theory of Higgs bundles is the *Hitchin fibration*. If $\mathscr{M}_{\mathrm{Higgs}}(r)$ denotes the moduli stack of rank r Higgs bundles, the Hitchin fibration is defined as follows

$$\begin{aligned} h : \mathscr{M}_{\mathrm{Higgs}}(r) &\longrightarrow \mathscr{B} = \bigoplus_{i=1}^{r} H^0(X, K_X^i) \\ (E, \varphi) &\longmapsto (b_1(\varphi), \dots, b_r(\varphi)), \end{aligned}$$

for $b_1(\varphi), \dots, b_r(\varphi)$ the coefficients of the characteristic polynomial

$$\det(\varphi - T\,\mathrm{id}) = T^r + b_1(\varphi)T^{r-1} + \cdots + b_r(\varphi).$$

More generally, one can consider the Hitchin fibration for the moduli stack $\mathscr{M}_{\mathrm{Higgs}}(G)$ of G-Higgs bundles, for G any reductive group of rank r, by taking the b_i to be generators of the ring of invariant polynomials $\mathbb{C}[\mathfrak{g}]^G$, for the adjoint action of G on $\mathfrak{g}$, and the *Hitchin base* $\mathscr{B} = \bigoplus_{i=1}^r H^0(X, K_X^{d_i})$, for $d_i = \deg b_i$. This map was introduced by Hitchin in his seminal paper [Hit87b]. The main result of his paper is that, for the vector bundle case, the generic fibre of an element $b \in \mathscr{B}$ is an abelian variety, more precisely, it is isomorphic to the Picard group of some ramified cover $Y_b \to X$ constructed from b and called the *spectral curve* associated to b. Hitchin also extends this to classical groups, where the fibres are isomorphic to some *Prym varieties* associated to the spectral curve.

The results of Hitchin can be generalized to the slightly more general case of *twisted* Higgs bundles, defined in the same way as Higgs bundles, but with the canonical bundle replaced by any other line bundle $L \to X$. This was considered by Beauville, Narasimhan and Ramanan [BNR89]. However, an important property of the case $L = K_X$ is that the dimension of the Hitchin base $\mathscr{B}$ is exactly half of the dimension of the moduli space, and thus the Hitchin fibration is an example of an *algebraically completely integrable system*.

Donagi and Gaitsgory [DG02] generalized the results of Hitchin to any reductive group G, by studying the *regular centralizers* and the *gerbe* structure of the Hitchin fibration. Their results are better understood using the "stacky point of view" of Ngô [Ng10]. In particular, Ngô used the properties of the Hitchin fibration to give a proof of the Fundamental Lemma of Langlands–Shelstad for the Lie algebras.

We shall explain here what is Ngô's "stacky" point of view of the Hitchin fibration, since it will be of great use in the following. One starts by considering the quotient stack $[\mathfrak{g}/G]$ of $\mathfrak{g}$ by the adjoint action of G. By construction, for any $\mathbb{C}$-scheme S, the groupoid $[\mathfrak{g}/G](S)$ of S-points of this stack is the groupoid of pairs (E, φ) with E a principal G-bundle on S and φ a section of the associated bundle $E(\mathfrak{g})$. Ngô studies the local properties of the Hitchin fibration by considering the invariant theory of the natural morphism $[\mathfrak{g}/G] \to \mathfrak{g} /\!\!/ G$ from the quotient stack to the GIT quotient. Indeed, the Hitchin fibration can be recovered from this as follows. The natural homothety action of $\mathbb{C}^*$ on $\mathfrak{g}$ commutes with the adjoint action of G, so one can consider the quotient stacks $[\mathfrak{g}/(G \times \mathbb{C}^*)]$ and $[(\mathfrak{g} /\!\!/ G)/\mathbb{C}^*]$. The stack of L-twisted G-Higgs bundles on X can be identified with the stack of maps $X \to [\mathfrak{g}/(G \times \mathbb{C}^*)]$ lying over the natural map $X \to [\operatorname{Spec} \mathbb{C}/\mathbb{C}^*] = \mathbb{B}\mathbb{C}^*$ defined by L, while the Hitchin base $\mathscr{B} = \bigoplus_{i=1}^r H^0(X, L^{d_i})$ is the stack of maps $X \to [(\mathfrak{g} /\!\!/ G)/\mathbb{C}^*]$ lying over that same map defined by L.

The stacky point of view allows to obtain the gerbe structure of the Hitchin fibration from the gerbe structure of $[\mathfrak{g}/G] \to \mathfrak{g} /\!\!/ G$. This is done as follows. One first considers the *centralizer group scheme* $I \to \mathfrak{g}$, with fibres $I_x = \{g \in G : \operatorname{Ad}_g(x) = x\}$. Ngô [Ng10] shows that, over the regular locus $\mathfrak{g}^{reg} \subset \mathfrak{g}$, the group scheme I descends to a group scheme J over $\mathfrak{g} /\!\!/ G$. This J is called the *regular centralizer* and one can prove that $[\mathfrak{g}/G] \to \mathfrak{g} /\!\!/ G$ has the structure of a gerbe banded by J (equivalently, of a $\mathbb{B}J$-torsor).

Langlands duality and mirror symmetry

There is a duality for Hitchin fibrations with Langlands dual structure groups by the results of Donagi and Pantev [DP12]. More precisely, we can consider G and $\check{G}$ two Langlands dual complex reductive groups (that is, G and $\check{G}$ have dual based root data), and their respective Hitchin fibrations $h_G : \mathscr{M}_{\mathrm{Higgs}}(G) \to \mathscr{B}(G)$ and $h_{\check{G}} : \mathscr{M}_{\mathrm{Higgs}}(\check{G}) \to \mathscr{B}(\check{G})$. Let $T \subset G$ be a maximal torus, that we identify with $\mathfrak{t}/X_*(T)$, $\mathfrak{t}$ being the Lie algebra of T and $X_*(T)$ the cocharacter lattice. We can consider $\check{T} \subset \check{G}$ the corresponding maximal torus, isomorphic to the dual torus of G

$$\check{T} \cong T^\vee = \mathfrak{t}^*/X^*(T),$$

where $X^*(T)$ denotes the character lattice. By making a choice of an invariant bilinear form in $\mathfrak{g}$, we can identify canonically $\mathfrak{t}$ and $\mathfrak{t}^*$, and also $X^*(T)$ and $X_*(T)$ and thus T and $\check{T}$.

As we explained above, for generic values $b \in \mathscr{B}(G)$ the Hitchin fibre $\mathscr{M}_b$ is an abelian variety. In fact, the gerbe structure that we explained above implies that these fibres are torsors under a certain Picard stack $\mathcal{P}_a$. The result that Donagi and Pantev obtain is that the choice of an invariant bilinear form induces an isomorphism of the Hitchin bases $\mathscr{B}(G) \to \mathscr{B}(\check{G})$, and that, generically, the Hitchin fibres are dual, with the duality given by a Fourier–Mukai transform on the Picard stacks $\mathcal{P}_a$. After the work of Kapustin and Witten [KW07], this can be interpreted in Physics terms as the "classical limit" of the geometric Langlands correspondence.

The duality of Hitchin systems is related with the physical theory of *mirror symmetry*. This theory gives a correspondence under which some objects called (B, A, A)-branes are matched to others called (B, B, B)-branes. These (B, A, A)-branes are elements of a certain Fukaya category, and are supported on submanifolds of the Hitchin moduli space which are Kähler with respect to the first complex structure I and Lagrangian with respect to J and K. On the other hand, (B, B, B)-branes are objects of a derived category of sheaves, supported on hyper-Kähler submanifolds.

THE MULTIPLICATIVE STORY

Singular monopoles and multiplicative Higgs bundles

The dimensional reduction of the self-dual Yang–Mills equations on $\mathbb{R}^4$ by translation in only one direction yields the *Bogomolny equation* on $\mathbb{R}^3$:

$$F_\nabla = *\nabla\Phi.$$

This equation is well defined over the product $Y = S^1 \times X$ of a circle and a Riemann surface and, more generally, one can consider the *Hermitian–Bogomolny equation*:

$$F_\nabla - *\nabla\Phi = iC\omega_X.$$

This is an equation for a pair (∇, Φ) formed by a Hermitian connection ∇ on a Hermitian vector bundle $\mathbb{E} \to S^1 \times X$ and $\Phi \in \Omega^0(X, \mathfrak{u}_h(\mathbb{E}))$ is a section of the bundle of endomorphisms preserving the Hermitian metric. The element C is a multiple of the identity in the group of automorphisms of $\mathbb{E}$. The same equation can be considered for any complex reductive group if one replaces vector bundles by principal G-bundles and instead of Hermitian metrics, one considers reductions of the structure group from G to a maximal compact subgroup $K \subset G$. The element C is then replaced by a central element of the Lie algebra $\mathfrak{k}$ of K.

We are interested in considering solutions to the Hermitian–Bogomolny equation with *Dirac-type singularities*. By definition, these solutions have a finite set of singular points $y_1, \dots, y_n \in Y$, and around each of these y_i one can find a small 3-dimensional ball over which the structure group of the bundle $\mathbb{E}$ reduces to a maximal torus and the pair (∇, Φ) is approximated by the image of the Dirac magnetic monopole through some cocharacter λ_i of G (understood as a homomorphism of compact Lie groups $\lambda_i : U(1) \to K$). Solutions to the Hermitian–Bogomolny equation with Dirac-type singularities are called *singular monopoles*.

The Hermitian–Bogomolny equation can be split in three equations

$$\begin{cases} F_{\nabla,X} - *\nabla_t \Phi = iC\omega_X, \\ [\nabla_X^{0,1}, \nabla_t - i\Phi dt] = 0, \\ [\nabla_X^{1,0}, \nabla_t + i\Phi dt] = 0. \end{cases}$$

Here, we have decomposed the connection in a X-part and a S^1-part as $\nabla = \nabla_X + \nabla_t$ and in turn we have split the X-part as $\nabla_X = \nabla_X^{1,0} + \nabla_X^{0,1}$. The first of these equations is called the *real equation* while the other two, which are equivalent, are the *complex equations*.

The two operators $\nabla_X^{0,1}$ and $\nabla_t - i\Phi dt$ define what is called a *mini-holomorphic structure* on $\mathbb{E}$; this is a term introduced by Mochizuki [Moc22]. On the one hand, the operator $\nabla_X^{0,1}$ induces a holomorphic structure on each of the fibres $\mathbb{E}_t = \mathbb{E}|_{\{e^{it}\}\times X}$, and we denote the resulting holomorphic bundle by E_t. On the other hand, taking parallel transport of the operator $\nabla_t - i\Phi dt$ along an interval $[t_1, t_2]$ yields a homomorphism

$$\varphi_{t_1,t_2} : \mathbb{E}_{t_1} \longrightarrow \mathbb{E}_{t_2}$$

called the *scattering map*. The complex equations imply that the scattering map defines a holomorphic homomorphism $E_{t_1} \to E_{t_2}$. More precisely, the scattering map is holomorphic when $[t_1, t_2] \times X$ does not contain any singular point, and meromorphic (with poles in the singular points) when it does.

In particular, if we take parallel transport along the whole circle S^1, we obtain a pair (E, φ), with $E = E_0 = E_{2\pi} \to X$ a holomorphic G-bundle and φ a meromorphic automorphism $\varphi : E \to E$. This is by definition a *multiplicative G-Higgs bundle*. If we restrict φ to a formal disc around a singular point $x_i \in X$ of φ, we obtain an element of the formal loop group $G(F)$, for $F = \mathbb{C}((z))$ the field of formal Laurent series in the formal variable z, which is well defined up to left and right multiplication by elements of the formal arc space $G(\mathcal{O})$, for $\mathcal{O} = \mathbb{C}[[z]]$ the ring of formal power series. The set of $G(\mathcal{O}) \times G(\mathcal{O})$ orbits of $G(F)$ is well known to be parametrized by dominant cocharacters of G (see Proposition 1.7.1). If (E, φ) is a multiplicative G-Higgs bundle arising from a singular monopole, then the cocharacter λ_i obtained from the scattering map around some singularity x_i is exactly the same cocharacter showing up in the Dirac type singularity.

Multiplicative Higgs bundles coming from an irreducible singular monopole must verify a certain *stability condition* (see Definition 3.2.12 for details). Reciprocally, it is a theorem of Charbonneau and Hurtubise [CH11], and of Smith [Smi16] in the general reductive group case, that every stable multiplicative Higgs bundle arises from a singular monopole.

The moduli space of monopoles can be constructed as a Kähler quotient and, when X has trivial canonical bundle K_X (which happens if and only if it has genus 1), it is in fact a hyper-Kähler quotient. The algebraic symplectic form on the moduli space coming from this hyper-Kähler structure was also constructed independently from the point of view of multiplicative Higgs bundles in the paper of Hurtubise and Markman [HM02].

The multiplicative Hitchin fibration

Let $\mathbf{x} = (x_1, \dots, x_n)$ be a tuple of points of the Riemann surface X and let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ a tuple of dominant cocharacters of G. We denote by $\mathscr{M}_{\mathbf{x},\boldsymbol{\lambda}}(G)$ the moduli stack of multiplicative G-Higgs bundles (E, φ) such that has a singularity at each x_i determined by the cocharacter λ_i. Consider the space

$$\mathscr{B}_{\mathbf{x},\boldsymbol{\lambda}}(G) = \bigoplus_{i=1}^{r} H^0(X, \mathscr{O}(\langle \omega_i, \boldsymbol{\lambda} \cdot \mathbf{x} \rangle)),$$

where $\omega_1, \dots, \omega_r$ are the fundamental weights of G and, if $\boldsymbol{\lambda} \cdot \mathbf{x} = \sum_{i=1}^{n} \lambda_i x_i$ is a divisor with values on dominant cocharacters, then $\langle \omega_i, \boldsymbol{\lambda} \cdot \mathbf{x} \rangle$ is the divisor

$$\langle \omega_i, \boldsymbol{\lambda} \cdot \mathbf{x} \rangle = \sum_{j=1}^{n} \langle \omega_i, \lambda_j \rangle x_j.$$

Let $\rho_i : G \to GL(V_i)$ denote the fundamental representation of G of highest weight ω_i, and let $b_i = \mathrm{tr}(\rho_i)$ be its trace. The polynomials $b_1, \dots, b_r$ generate the ring of invariant polynomials $\mathbb{C}[G]^G$ under the adjoint action of G on itself. The *multiplicative Hitchin fibration* is now defined as the map

$$\begin{aligned} \mathscr{M}_{\mathbf{x},\boldsymbol{\lambda}}(G) &\longrightarrow \mathscr{B}_{\mathbf{x},\boldsymbol{\lambda}}(G) \\ (E, \varphi) &\longmapsto (b_1(\varphi), \dots, b_r(\varphi)). \end{aligned}$$

The multiplicative Hitchin fibration was originally studied by Hurtubise and Markman [HM02], who in particular prove that, when X has genus 1, it defines an algebraically completely integrable system.

The monoid point of view

A very convenient point of view on the study of multiplicative Higgs bundles is to consider them in the context of the theory of *reductive monoids*. A reductive monoid is an algebraic monoid M over $\mathbb{C}$ (that is, a monoid object in the category of $\mathbb{C}$-schemes) such that its unit group $M^\times$ is a reductive group. A prototypical example is the monoid of $n \times n$ matrices Mat_n, which has as unit group the general linear group, $\mathrm{Mat}_n^\times = \mathrm{GL}_n$. The point of view of monoids allows (when G is semisimple) to see the multiplicative Hitchin fibration in a stacky framework very similar to Ngô's stacky point of view of the Hitchin fibration.

Suppose that G is semisimple simply-connected and let M be a monoid such that the derived group of $M^\times$ is equal to G. The GIT quotient $\mathbb{A}_M = M /\!\!/ (G \times G)$ is a toric Z-variety, for $Z = Z^0_{M^\times}$ the neutral connected component of the centre of the unit group $M^\times$, called the *abelianization* of the monoid M. A monoid is *very flat* if the quotient map $M \to \mathbb{A}_M$ is flat, dominant and with integral fibres. A morphism of monoids $M_1 \to M_2$ is *excellent* if the commutative square induced with the abelianizations is Cartesian (in other words, if $M_1 = M_2 \times_{\mathbb{A}_{M_2}} \mathbb{A}_{M_1}$). The category of very flat monoids with semisimple part G and with excellent morphisms has an universal object called the Vinberg monoid or the enveloping monoid of G, and denoted by $\mathrm{Env}(G)$. The unit group of $\mathrm{Env}(G)$ is the group $G_+ = (G \times T)/Z$, for $T \subset G$ a maximal torus, so that it has centre $Z_+ \cong T^{\mathrm{ad}}$, and its abelianization is the T^{ad}-toric variety $\mathbb{A}_{\mathrm{Env}(G)}$ determined by the weight semigroup

$$P_+(\mathbb{A}_{\mathrm{Env}(G)}) = \mathbb{Z}_+ \{\alpha_1, \ldots, \alpha_r\},$$

for $\alpha_1, \ldots, \alpha_r$ the simple roots of T.

Note that the S-points of the quotient stack $[\mathbb{A}_{\mathrm{Env}(G)}/T^{\mathrm{ad}}]$ consist of pairs (L, s) formed by a T^{ad}-torsor L over S and a section s of the associated bundle $L(\mathbb{A}_{\mathrm{Env}(G)})$. A pair $(\mathbf{x}, \boldsymbol{\lambda})$ formed by a tuple of n points of X and a tuple of n dominant cocharacters naturally defines an X-point of $[\mathbb{A}_{\mathrm{Env}(G)}/T^{\mathrm{ad}}]$, by taking L to be the T^{ad}-bundle defined by taking each λ_i as its transition function on a disc around x_i and s to be the natural nonvanishing section of

$$L(\mathbb{A}_{\mathrm{Env}(G)}) = \bigoplus_{i=1}^{r} \mathcal{O}\left(\sum_{j=1}^{n} \langle \alpha_i, \lambda_j \rangle x_i\right).$$

Indeed, this nonvanishing section exists as a consequence of the characters λ_j being dominant. The moduli stack of multiplicative G-Higgs bundles is then recovered as the stack of maps $X \to [\mathrm{Env}(G)/(G \times T^{\mathrm{ad}})]$ lying over the natural map $X \to [\mathbb{A}_{\mathrm{Env}(G)}/T^{\mathrm{ad}}]$ induced by the pair $(\mathbf{x}, \boldsymbol{\lambda})$.

INVOLUTIONS

Involutions, symmetric varieties and real forms

In this dissertation we are mainly concerned with the interplay of multiplicative Higgs bundles and monopoles with holomorphic involutions θ of the structure group G. By an *involution* of G we mean an automorphism $\theta \in \mathrm{Aut}(G)$ of order 2 (that is, $\theta^2 = \mathrm{id}$). Associated to an involution, we can consider the fixed point subgroup

$$G^\theta = \{g \in G : \theta(g) = g\}.$$

Moreover, the involution θ induces the *Cartan decomposition* of the Lie algebra $\mathfrak{g}$ of G as
$$\mathfrak{g} = \mathfrak{g}^\theta \oplus \mathfrak{m}^\theta,$$
for $\mathfrak{g}^\theta$ and $\mathfrak{m}^\theta$ the $+1$ and -1 eigenspaces of θ, respectively. The space $\mathfrak{g}^\theta$ is indeed the Lie algebra of G^θ, while $\mathfrak{m}^\theta$ is the tangent space at the identity of the symmetric variety G/G^θ.

More generally, by a *symmetric subgroup associated to* θ, we mean a closed subgroup $H \subset G$ such that $G_0^\theta \subset H \subset G_\theta$, where $G_0^\theta = (G^\theta)^0$ is the neutral connected component of G^θ and $G_\theta = \left\{g \in G : g\theta(g)^{-1} \in Z_G\right\}$ is equal to the normalizer of G^θ. Here, Z_G denotes the *centre* of G. The homogeneous space G/H, where $H \subset G$ is a symmetric subgroup, is called a *symmetric variety*.

The symmetric variety G/G^θ can be realized inside the group G as the subvariety $M^\theta = \left\{g\theta(g)^{-1} : g \in G\right\}$. More generally, we can consider the θ-twisted conjugation action $g *_\theta x = gx\theta(g)^{-1}$ of G on itself. The θ-twisted orbits are homogeneous spaces of the form G/G^{θ_s}, for $\theta_s = \mathrm{Int}_s \circ \theta$, and Int_s the inner automorphism given by conjugation by s. The automorphism θ_s will only be an involution if it belongs to the subset
$$S_\theta = \{s \in G : s\theta(s) \in Z_G\}.$$
This allows to identify the subset S_θ with the elements of the class $a \in \mathrm{Out}_2(G)$ of θ in the group $\mathrm{Out}_2(G) = \mathrm{Aut}_2(G)/\mathrm{Int}(G)$.

Moreover, the groups G^θ and $G^{\theta'}$ (and thus the associated symmetric varieties) are identified if θ and θ' are conjugate by an inner automorphism. This induces a further relation $\sim$ in the group of involutions $\mathrm{Aut}_2(G)$, and the restriction of the natural map $\mathrm{Aut}_2(G) \to \mathrm{Out}_2(G)$ to $\mathrm{Aut}_2(G)/\sim$ is the *clique map* $\mathrm{cl} : \mathrm{Aut}_2(G)/\sim \to \mathrm{Out}_2(G)$ of García-Prada and Ramanan [GPR19]. The fibre of an element $a \in \mathrm{Out}_2(G)$ by the clique map can be identified with the quotient
$$\mathrm{cl}^{-1}(a) = S_\theta/(G \times Z_G)$$
of S_θ (for any representative θ of a) by the action of $G \times Z_G$ defined as
$$(g, z) * x = zg *_\theta x.$$

It is a classical result of Cartan [Car14], that involutions and real forms of G can be identified. Indeed, for any involution θ there exists a maximal compact real form K with associated conjugation σ_K of G commuting with θ, and one can consider the real form $\sigma_\theta = \sigma_K \circ \theta$. Reciprocally, given any conugation σ there exists some involution θ_σ such that $\sigma \circ \theta_\sigma$ is the conjugation of a maximal compact real form.

Higgs bundles for real forms

Let $\theta \in \mathrm{Aut}_2(G)$ be an involution of G. By a (G, θ)-*Higgs bundle* we mean a pair (E, φ), where $E \to X$ is a principal G^θ-bundle and φ is a section of the bundle $E(\mathfrak{m}^\theta) \otimes K_X$. Here, $E(\mathfrak{m}^\theta)$ is the bundle associated to the natural action of G^θ on $\mathfrak{m}^\theta$ induced by the adjoint action of G on $\mathfrak{g}$.

In the literature, (G,θ)-Higgs bundles usually receive the name of $G_{\mathbb{R}}$-*Higgs bundles*, where $G_{\mathbb{R}}$ is the real form of G associated to θ. The reason behind this nomenclature is that under the nonabelian Hodge correspondence these objects are known to yield representations of the fundamental group of X factoring through the real group $G_{\mathbb{R}}$ (see [GP20] for further references).

In the work of García-Prada and Ramanan [GPR19], it is proven that (G,θ)-Higgs bundles form the fixed point locus of the natural involution on the moduli space of G-Higgs bundles given by

$$(E,\varphi) \mapsto (\theta(E), -\theta(\varphi)).$$

The involution ι_-^θ that we study in this dissertation is a "multiplicative version" of this one. An easy consequence of this fact is that (G,θ)-Higgs bundles form a submanifold of the Hitchin moduli space which is Kähler with respect to the first complex structure I and Lagrangian with respect to the other two.

It is suggested that Higgs bundles for real forms $G_{\mathbb{R}}$ of G give (B,A,A)-branes in the moduli space of G-Higgs bundles corresponding to $\check{G}_{G_{\mathbb{R}}} \subset \check{G}$, the dual group of the real form $G_{\mathbb{R}}$ introduced by Nadler [Nad05]. This conjecture arises from a gauge-theoretical description of the construction of the Nadler group, given by Gaiotto and Witten [GW09]. See Section 1.4 for more information about the dual group.

The invariant theory of the action of G^θ on $\mathfrak{m}^\theta$ was studied by Kostant and Rallis [KR71]. Their work gives a generalization of the Chevalley isomorphism by proving that the GIT quotient $\mathfrak{m}^\theta /\!\!/ G^\theta$ is isomorphic to $\mathfrak{a}/W_\theta$, where $\mathfrak{a} \subset \mathfrak{m}^\theta$ is a maximal abelian subalgebra and W_θ is the *little Weyl group* associated to the involution θ (see Proposition 1.3.11). This allows to define a version of the Hitchin fibration for (G,θ)-Higgs bundles, also known as the Hitchin fibration for real groups or the Hitchin fibration for symmetric pairs. This version of the Hitchin fibration was introduced in the thesis of Peón-Nieto [PN13], and later explored further in her work with García-Prada and Ramanan [GPPNR18, GPPN23]. A complete description of the Hitchin fibration for symmetric pairs will appear in forthcoming work of Morrissey and Hameister [HM] (the interested reader can consult Hameister's talk [Ham23]).

Involutions on the space of multiplicative Higgs bundles

An involution $\theta \in \mathrm{Aut}_2(G)$ induces involutions on the space of multiplicative Higgs bundles in a very similar way as it does in the space of Higgs bundles. More precisely, for any multiplicative G-Higgs bundle (E,φ) one can consider

$$\iota_\pm^\theta(E,\varphi) = (\theta(E), \theta(\varphi)^{\pm 1}).$$

At the level of isomorphism classes, the involutions $\iota_\pm^\theta$ only depend on $a \in \mathrm{Out}_2(G)$ the class of θ in $\mathrm{Out}_2(G)$, and thus it makes sense to denote $\iota_\pm^a$. At the level of mini-holomorphic bundles on $Y = S^1 \times X$, one can show that these involutions correspond to

$$\iota_\pm^\theta(\mathcal{E}) = \zeta_\pm^* \theta(\mathcal{E})$$

where $\mathcal{E} \to Y$ denotes the mini-holomorphic bundle corresponding to (E, φ), and $\zeta_\pm : Y \to Y$ are the involutions

$$\begin{aligned} \zeta_\pm : S^1 \times X &\longrightarrow S^1 \times X \\ (e^{it}, x) &\longmapsto (e^{\pm it}, x). \end{aligned}$$

The fixed points of the involution ι^a_- include *multiplicative* (G, θ)*-Higgs bundles*, which we recall that are pairs (E, φ) with $E \to X$ a holomorphic principal G^θ-bundle and φ a meromorphic section of the associated bundle of symmetric varieties $E(G/G^\theta)$. This makes sense because multiplicative (G, θ)-Higgs bundles extend naturally to multiplicative G-Higgs bundles, since the left multiplication action of G on G/G^θ corresponds to the θ-twisted conjugation action on M^θ and thus G^θ acts on M^θ by conjugation.

The local singularities of multiplicative (G, θ)-Higgs bundles can be prescribed in a similar way as for multiplicative G-Higgs bundles. If we restrict the section φ to a formal disc around a singular point $x_i \in X$, we obtain an element of the formal loop space $(G/G^\theta)(F)$ which is well defined up to multiplication by elements of $G(\mathcal{O})$. The $G(\mathcal{O})$ orbits of $(G/G^\theta)(F)$ are known to be parametrized by antidominant cocharacters of the torus $A_{G^\theta} = A/(A \cap G^\theta)$, where A is a *maximal* θ*-split torus* (that is, A is maximal among all tori of G such that $\theta(a) = a^{-1}$ for all $a \in A$).

However, apart from multiplicative (G, θ)-Higgs bundles, other objects appear as fixed points of $\iota^a_\pm$. This is an essential difference between the "additive" case and the multiplicative case, which stems from the fact that, although $\mathfrak{m}^\theta$ is precisely the set of anti-fixed points of $\mathfrak{g}$ under the involution induced by θ, the set $S^\theta = \{s \in G : \theta(s) = s^{-1}\}$ of anti-fixed points contains the symmetric variety $M^\theta \cong G/G^\theta$, but also other components, isomorphic to the symmetric varieties G/G^{θ_s}, for $s \in S^\theta$ and $\theta_s = s\theta(-)s^{-1}$. The other objects are thus pairs (E, φ) with E a G^θ-bundle and φ a meromorphic section of the bundle $E(G/G^{\theta_s})$ with G^θ acting on G/G^{θ_s} by left multiplication. The possible values of s that can appear as fixed points are restricted by the value of cocharacters at the singularities (see Corollary 2.3.9).

As in the case of (G, θ)-Higgs bundles, a Hitchin map can be constructed for multiplicative (G, θ)-Higgs bundles. The construction relies on Richardson's [Ric82b] study of the invariant theory of the action of G^θ on M^θ. Richardson proves that the GIT quotient $M^\theta /\!\!/ G^\theta$ is isomorphic to A/W_θ, for A a maximal θ-split torus, and W_θ the little Weyl group associated to θ.

The point of view of symmetric embeddings

The "monoid point of view" of multiplicative G-Higgs bundles can be generalized to the theory of multiplicative (G, θ)-Higgs bundles by considering *equivariant embeddings of symmetric varieties*. For short, we call these kind of embeddings *symmetric embeddings*.

We start by taking G to be semisimple simply-connected and considering the symmetric variety G/G^θ. Suppose that Σ is an equivariant embedding of a symmetric variety O_Σ such that its *semisimple part* is equal to G/G^θ. By definition, the semisimple part of a symmetric variety $O_\Sigma = G_1/H_1$ is the homogeneous space $G_1'/(H_1 \cap G_1')$, where G_1' is the derived group of G_1. We can define the *abelianization* of the symmetric embedding Σ as the GIT quotient $\mathbb{A}_\Sigma = \Sigma /\!\!/ G$, which is a toric A_Σ-variety, for $A_\Sigma = O_\Sigma/G$. As with monoids, we say that a symmetric embedding Σ is *very flat* if the quotient $\Sigma \to \mathbb{A}_\Sigma$ is flat, dominant and with integral fibres. A morphism of symmetric embeddings $\Sigma_1 \to \Sigma_2$ is *excellent* if $\Sigma_1 = \Sigma_2 \times_{\mathbb{A}_{\Sigma_2}} \mathbb{A}_{\Sigma_1}$. The category of very flat symmetric embeddings with semisimple part G/G^θ and with excellent morphisms has an universal object called the *Guay embedding* or the *enveloping embedding* of G/G^θ, and denoted by $\mathrm{Env}(G/G^\theta)$. The torus $A_{\mathrm{Env}(G/G^\theta)}$ is equal to $A_{G_\theta} := A/(A \cap G_\theta)$, for $A \subset G$ a maximal θ-split torus, and the abelianization $\mathbb{A}_{\mathrm{Env}(G/G^\theta)}$ is the A_{G^θ}-toric variety determined by the weight semigroup

$$P_+(\mathbb{A}_{\mathrm{Env}(G/G^\theta)}) = -\mathbb{Z}_+ \{\bar{\alpha}_1, \dots, \bar{\alpha}_l\},$$

for $\bar{\alpha}_1, \dots, \bar{\alpha}_l$ the simple restricted roots associated to θ (see Section 1.3).

Now, the S-points of the quotient stack $[\mathbb{A}_{\mathrm{Env}(G/G^\theta)}/A_{G_\theta}]$ consist of pairs (L, s) formed by a A_{G_θ}-torsor L over S and a section s of the associated bundle $L(\mathbb{A}_{\mathrm{Env}(G/G^\theta)})$. A pair $(\mathbf{x}, \boldsymbol{\lambda})$ formed by a tuple of n points of X and n antidominant cocharacters of A_{G^θ} naturally defines an X-point of $[\mathbb{A}_{\mathrm{Env}(G/G^\theta)}/A_{G_\theta}]$, by taking L to be the A_{G_θ}-bundle defined by taking the λ_i as transition functions and s the natural nonvanishing section of

$$L(\mathbb{A}_{\mathrm{Env}(G/G^\theta)}) = \bigoplus_{i=1}^{l} \mathcal{O}\left(\sum_{j=1}^{n} \langle -\bar{\alpha}_i, \lambda_j \rangle x_i\right),$$

which exists as a consequence of the λ_j being antidominant. The moduli stack of multiplicative (G, θ)-Higgs bundles is then recovered as the stack of maps $X \to [\mathrm{Env}(G/G^\theta)/(G^\theta \times A_{G_\theta})]$ lying over the natural map $X \to [\mathbb{A}_{\mathrm{Env}(G/G^\theta)}/A_{G_\theta}]$ induced by the pair $(\mathbf{x}, \boldsymbol{\lambda})$.

OUTLINE AND MAIN RESULTS

This dissertation consists of three chapters, plus this introduction and a final digression about possible further directions that can be pursued based on the results of this thesis.

Chapter 1 covers the theory of symmetric varieties and their embeddings. The purpose of the chapter is essentially giving some of the preliminary notions needed to understand the next chapters, specially Chapter 2. However, the chapter also contains some original results concerning the invariant theory of symmetric embeddings and their loop parametrization. We begin by reviewing some general facts about reductive groups with involutions and explain the related points of view concerning real forms and nonabelian group cohomology. After that, we introduce the notions of θ-split tori and θ-split parabolic subgroups and review the construction of the restricted root system and its associated root and weight lattices. We continue by reviewing the theory of equivariant embeddings of symmetric varieties, originally developed by Vust [Vus90] as a particular case of the more general theory of spherical varieties of Luna and Vust [LV83]. We review some basic notions about affine spherical varieties and describe the weight lattice and the weight semigroup of a symmetric variety. We also recall the definition and the main properties of the dual group of a symmetric variety. In Table 1.2 we summarize some information about the involutions of the simple groups.

Chapter 1 goes on by explaining the construction and main properties of the *wonderful compactification* of a symmetric variety, being of particular interest the local structure theorem of De Concini and Procesi. We continue by explaining Guay's theory of very flat embeddings of symmetric varieties [Gua01]. In particular, we review the notion of abelianization of an embedding, Guay's characterization of very flat embeddings, and the construction and main properties of the Guay enveloping embedding. We finish that section by proving some original result, Proposition 1.6.15, concerning the invariant theory of a symmetric embedding. In the section following that one, we study the *loop parametrization* of a symmetric variety and its embeddings, by what we mean that we study their spaces of formal loops and describe the orbits of the formal arc group in them. In particular, in Proposition 1.7.7, we generalize Proposition 1.8.6, which is a result of J. Chi [Chi22]. We finish the chapter by reviewing the theory of reductive monoids as a particular case of Guay's theory of symmetric embeddings.

Chapter 2 is centered around multiplicative Higgs bundles over the compact Riemann surface X. We start with a preliminary section reviewing the theory of multiplicative Higgs bundles as introduced originally by Hurtubise and Markman [HM02], on the one hand, and as independently developed by Bouthier, J. Chi and G. Wang [Bou15, BC18, Bou17, Chi22, Wan23] using the point of view of reductive monoids, on the other. In particular, we start by explaining a very general point of view around the Hitchin map and its analogues and generalizations suggested by Morrissey and Ngô [MN] (the interested reader may consult Ngô's talk [Ng23]), and then go on to explain two equivalent points of view around the multiplicative Hitchin fibration. We also review Hurtubise and Markman's arguments for the existence of a moduli space of simple multiplicative Higgs bundles, and their description of the tangent space and, when X has genus 1, of the holomorphic symplectic form in the moduli space.

We continue Chapter 2 with two sections of original results concerning our generalization of the theory of multiplicative Higgs bundles from reductive groups to symmetric varieties. In Section 2.2 we define multiplicative (G, θ)-Higgs bundles and the Hitchin map associated to them, and show that it is equivalent to a natural Hitchin map arising from the theory of symmetric embeddings. This is the content of our Theorem 2.2.5 which is a generalization of the result of Bouthier, Chi and Wang (our Theorem 2.1.9). We also show in Section 2.2 how multiplicative (G, θ)-Higgs bundles can be realized inside the moduli space of multiplicative G-Higgs bundles. The last section of Chapter 2, Section 2.3 gives a full description of the involutions $\iota_{\pm}^{\theta} : (E, \varphi) \mapsto (\theta(E), \theta(\varphi)^{\pm 1})$ on the moduli space of multiplicative G-Higgs bundles and their fixed points. The main results of this part are Theorem 2.3.4 and Corollary 2.3.9. We finish by describing the interplay of the involutions $\iota_{\pm}^{\theta}$ with the Hurtubise–Markman symplectic form in Theorem 2.3.10.

Chapter 3 studies the counterpart of the problem studied in the previous Chapter 2 from the point of view of mini-holomorphic bundles and monopoles. We begin with a section introducing the concept of a *mini-holomorphic principal* G-*bundle* over $Y = S^1 \times X$ the product of a circle and a compact Riemann surface. This concept is a generalization of the notion of mini-holomorphic vector bundle as introduced by Mochizuki [Moc22]. We also introduce the associated notions of the Chern pair, Dirac-type singularities and the scattering map, and explain the equivalence between mini-holomorphic bundles and multiplicative Higgs bundles. Mini-holomorphic principal bundles were already considered in the work of Smith [Smi16] (what he calls "holomorphic structures"). However, we use here a different, more intrisic approach (not relying on representations nor on covariant derivatives on associated vector bundles) that, to our best knowledge, is not already present in the literature.

The next sections of Chapter 3 are dedicated to study monopoles. First, we introduce the Hermitian–Bogomolny equation and the notion of singular monopole, and review the results of Charbonneau–Hurtubise [CH11] and Smith [Smi16] relating monopoles with polystable mini-holomorphic bundles. We also explain the construction of the moduli space of monopoles as a Kähler quotient, and as a hyper-Kähler quotient when X has genus 1. Finally, in Section 2.3.9 we describe the involutions $\iota_{\pm}^{\theta}$ and their fixed points from the point of view of mini-holomorphic bundles, and recover the results of Section 2.3 in Propositions 3.4.3 and 3.4.4. The section ends by describing the involutions entirely in terms of singular monopoles, and describing the interplay of the involutions with the hyper-Kähler structure of the moduli space of monopoles, when X has genus 1.

The dissertation ends with an exploration of different further directions of research that could ramify from this thesis. We begin with several conjectures (Conjectures 1, 2 and 3) about possible analogues of the results of Donagi–Pantev regarding duality of Hitchin systems to the multiplicative Hitchin fibration. The proof of these conjectures is the content of forthcoming joint work with Benedict Morrissey [GM]. We continue by suggesting some problems and questions regarding the description of the Hitchin map for multiplicative (G, θ)-Higgs bundles through the study of the *regular quotients*, in the sense of Morrissey and Ngô [MN, Ng23] of the symmetric variety G/G^{θ} and of the enveloping embedding $\mathrm{Env}(G/G^{\theta})$ by the action of G^{θ}. In particular, we pose the question of the existence of a cross-section of the GIT quotient $G/G^{\theta} \to (G/G^{\theta}) /\!\!/ G^{\theta}$ generalizing the Kostant–Rallis section [KR71] and the Steinberg cross-section [Ste65]. In Questions 2 and 3 we propose the study of a possible generalization of Guay's theory of very flat embeddings of symmetric varieties to more general spherical varieties. This could in turn give a generalization of some of the result of this thesis to spherical varieties. In Problem 2, we suggest the existence of a different gauge-theoretical description of multiplicative Higgs bundles which, as opposed to the theory of singular monopoles, is defined intrinsically for the Riemann surface X and does not need to make use of the product with cirle S^1, at least explicitly. We expect that this could be done by studying Kähler structures on very flat monoids and their associated moment maps for the action of a maximal compact subgroup K of G, and by applying the theory of pairs developed by Mundet i Riera [MiR00]. We finish the section of further directions by proposing the study of a "de Rham side" in the moduli space of singular monopoles. Another interesting possible direction, that we do not include in this section since we leave it completely unexplored, is the possible application of the result of this thesis to the pursue of some "relative" version (in the sense of Ben-Zvi–Sakellaridis–Venkatesh [BZSV23]) of the Fundamental Lemma of Langlands–Shelstad for symmetric varieties, generalizing the thesis of G. Wang [Wan23].

Out of convenience, in Chapters 1 and 2 we work over the algebraic setting, instead of over the (equivalent) holomorphic setting. That is, we consider G to be a reductive algebraic group over $\mathbb{C}$ and X to be a smooth projective complex curve instead of a complex reductive group and a compact Riemann surface, respectively, and consider algebraic (instead of holomorphic) principal bundles over X. The equivalence of both settings is a consequence of Serre's GAGA [Ser56]. It follows from the fact that all the arguments of these chapters are algebraic, that everything in them (except the comments concerning real forms) remains true if we replace $\mathbb{C}$ by any algebraically closed field of characteristic 0.

INTRODUCCIÓN (EN ESPAÑOL)

UNA VISIÓN PANORÁMICA

Un G-*fibrado de Higgs multiplicativo* en X es un par (E, φ), donde E es un G-fibrado principal holomorfo sobre X y φ es una sección meromorfa del fibrado adjunto de grupos $E(G) := E \times_G G$. En efecto, esta es una versión "multiplicativa"de un G-*fibrado de Higgs* (torcido) en X, que es un par (E, φ) con E un G-fibrado principal holomorfo sobre X y φ una sección del fibrado adjunto de álgebras de Lie $E(\mathfrak{g})$, tensorizado por un fibrado de línea L sobre X. En el caso multiplicativo, en vez de fijarse el fibrado de línea L, se controla la singularidad en cada punto singular x fijando un cocaracter dominante de G.

Charbonneau y Hurtubise [CH11] y, posteriormente, Smith [Smi16] han definido condiciones de *estabilidad* para los G-fibrados de Higgs multiplicativos y han demostrado una correspondencia de *tipo Hitchin–Kobayashi* entre G-fibrados de Higgs multiplicativos estables y K-*monopolos singulares* sobre la 3-variedad $S^1 \times X$, donde S^1 denota la circunferencia unidad y $K \subset G$ is a es un subgrupo compacto maximal. Un K-monopolo singular sobre $S^1 \times X$ es una tripleta $(\mathbb{E}, \nabla, \Phi)$, con $\mathbb{E}$ un K-fibrado principal sobre $S^1 \times X$, ∇ una K-conexión en $\mathbb{E}$ y Φ una sección del fibrado adjunto $\mathbb{E}(\mathfrak{k})$, de tal manera que

1. $(\mathbb{E}, \nabla, \Phi)$ puede ser localmente aproximado por un T_K-monopolo de Dirac estándar en $\mathbb{R}^3$, para $T_K \subset K$ un toro maximal y tal que
2. resuelve la *ecuación de Hermite–Bogomolny*

$$F_\nabla - *\nabla\Phi = iC\omega_X,$$

para un cierto elemento central $C \in Z(\mathfrak{k})$. Aquí, F_∇ denota la curvatura de ∇, y $*$ es el operador estrella de Hodge.

Consideremos ahora la involución $\theta \in \mathrm{Aut}_2(G)$. Usando esta involución, podemos construir dos G-fibrados de Higgs multiplicativos diferentes empezando con uno dado (E, φ), en efecto, ponemos

$$\iota_+^\theta(E, \varphi) = (\theta(E), \theta(\varphi)) \quad \text{y} \quad \iota_-^\theta(E, \varphi) = (\theta(E), \theta(\varphi)^{-1}).$$

Nótese que aplicando el mismo proceso otra vez, recuperamos el par (E, φ) con el que empezamos. El propósito principal de esta tesis es describir los puntos fijos de estas involuciones. Por supuesto, pasando por la correspondencia de Charbonneau–Hurtubise–Smith, podemos considerar las involuciones correspondientes en el espacio de móduli de monopolos. En este documento describimos con precisión cuál es esta involución en el espacio de móduli de monopolos y damos una descripción equivalente de los puntos fijos desde el punto de vista de los monopolos.

Entre las diferentes clases de objetos que aparecen como puntos fijos de la involución ι_-^θ, hay una de especial interés, que es la formada por lo que denominamos (G,θ)-*fibrados de Higgs multiplicativos*. Éstos son una generalización de los G-fibrados de Higgs multiplicativos. Un (G,θ)-fibrado de Higgs multiplicativo en X es un par (E,φ) con $E\to X$ un G^θ-fibrado principal holomorfo y φ una sección meromorfa del fibrado asociado de variedades simétricas $E(G/G^\theta)$.

Cuando G es semisimple, Frenkel y Ngô [FN11] sugieren la idea de que, si (E,φ) es un G-fibrado de Higgs multiplicativo, entonces se puede entender la sección φ como una aplicación holomorfa (en vez de meromorfa), si se extiende a un *monoide reductivo muy plano* tal que el grupo derivado de su grupo de unidades sea igual a G. Esta idea ha sido crucial en los trabajos de Bouthier y J. Chi dedicados a estudiar análogos de las fibras de Springer afines [Bou15, Bou17, BC18, Chi22], y en última instancia ha conducido a la demostración del Lema Fundamental de Langlands–Shelstad por G. Wang [Wan23], en el espíritu del trabajo de Ngô sobre el Lema Fundamental para las álgebras de Lie [Ng10].

Aquí, generalizamos este «punto de vista de los monoides» y algunos de los resultados de Bouthier, Chi y Wang al contexto de los (G,θ)-fibrados de Higgs multiplicativos. Para hacer esto, usamos la teoría de Guay de las *inmersiones muy planas* de variedades simétricas [Gua01] y mostramos que la sección φ de un (G,θ)-fibrado de Higgs multiplicativo se puede extender a una inmersión muy plana Σ de la variedad simétrica G/G^θ. Motivados por los resultados de Wang, esperamos que nuestra teoría de (G,θ)-fibrados de Higgs multiplicativos establezca el marco correcto para una demostración de una generalización del Lema Fundamental para los grupos a las variedades simétricas, en el espíritu del Programa de Langlands Relativo de Ben-Zvi–Sakellaridis–Venkatesh [BZSV23]. En esta tesis, esta dirección la dejamos completamente inexplorada.

En el contexto de los fibrados de Higgs multiplicativos existe un análogo de la fibración de Hitchin [Hit87b]. Dicho análogo fue originalmente considerado por Hurtubise–Markman [HM02]. La fibración de Hitchin multiplicativa también ha sido estudiada desde este «punto de vista de los monoides» en los trabajos de Frenkel–Ngô, Bouthier, Chi y Wang y también se puede generalizar a los (G,θ)-fibrados de Higgs multiplicativos. El estudio de las fibras y las simetrías de la aplicación de Hitchin para (G,θ)-fibrados de Higgs no se considera aquí, aunque incluimos algunos comentarios al respecto en el apartado sobre direcciones futuras al final de la memoria.

Los (G,θ)-fibrados de Higgs multiplicativos son el análogo multiplicativo de los fibrados de Higgs «para grupos reales», que bajo la *correspondencia de Hodge no abeliana* se asocian a representaciones del grupo fundamental de X en la forma real de G asociada a θ. Por el momento no se conoce una «versión multiplicativa» de la correspondencia de Hodge no abeliana, pero sí se espera que exista al menos en el caso en el que X tiene género 1, ya que en tal caso el espacio de móduli de monopolos es naturalmente hiperkähleriano. Nuestra descripción de los monopolos fijos por la involución ι_-^θ debería explicar cuáles son los objetos que les corresponden en este «lado de de Rham» conjetural.

Es bien sabido que la fibración de Hitchin exhibe una forma de *dualidad de Langlands y simetría* mirror. Con más precisión, el trabajo de Donagi y Pantev [DP12] muestra que las bases de Hitchin asociadas a un grupo reductivo G y a su dual de Langlands $\check{G}$ se identifican naturalmente mediante la elección de una forma bilinear invariante, mientras que las fibras de Hitchin son variedades abelianas duales. Se conjetura que se puede hacer una construcción de (B, B, B)-branas en el espacio de móduli de $\check{G}$-fibrados de Higgs a partir de (B, A, A)-branas en el espacio de G-fibrados de Higgs mediante la aplicación de la transformada de Fourier–Mukai sobre las fibras de Hitchin. Los fibrados de Higgs para formas reales $G_{\mathbb{R}}$ definen el soporte de una (B, A, A)-brana que se conjetura dual al espacio de móduli de $\check{G}_{G_{\mathbb{R}}}$-fibrados de Higgs, para $\check{G}_{G_{\mathbb{R}}}$ el grupo dual de Nadler de la forma real $G_{\mathbb{R}}$. En este documento mostramos cómo los (G, θ)-fibrados de Higgs multiplicativos (y más generalmente, los puntos fijos de ι_-^{θ}) forman el soporte de una (B, A, A)-brana en el espacio de móduli de G-fibrados de Higgs multiplicativos (cuando X tiene género 1). Conjeturamos que en el caso multiplicativo se dan resultados análogos a los de Donagi–Pantev, y que la (B, B, B)-brana dual correspondiente a los (G, θ)-fibrados de Higgs multiplicativos está formada por $\check{G}_{G_{\mathbb{R}}}$-fibrados de Higgs multiplicativos.

LA HISTORIA CLÁSICA

Fibrados de Higgs y el espacio de móduli de Hitchin

La teoría de los fibrados de Higgs fue iniciada por Hitchin [Hit87a] en el contexto del estudio de la reducción dimensional de las ecuaciones de Yang–Mills autoduales en $\mathbb{R}^4$ por la acción de las traslaciones en dos direcciones. Las ecuaciones resultantes en $\mathbb{R}^2$ son conformemente invariantes, y por tanto pueden considerarse sobre una superficie de Riemann compacta X. Así, se obtienen las que se conocen como las *ecuaciones de Hitchin*:

$$\begin{cases} F_\nabla + [\varphi, \varphi^\dagger] = -2\pi i \frac{d}{r} \operatorname{id}_{\mathbb{E}} \omega_X, \\ \nabla^{0,1}\varphi = 0. \end{cases}$$

Éstas son ecuaciones para un par (∇, φ), donde ∇ es una conexión hermítica en un fibrado vectorial hermítico $\mathbb{E} \to X$ de rango r y grado d, y $\varphi \in \Omega^{1,0}(X, \operatorname{End} \mathbb{E})$ es una $(1, 0)$-forma con valores en el fibrado de endomorfismos $\operatorname{End} \mathbb{E}$. La notación $[\varphi, \varphi^\dagger]$ indica que se toma el conmutador en la parte matricial, y el producto *wedge* en la parte de forma. Empleamos ω_X para denotar la forma de área en X, que escogemos normalizada a área total 1. El operador ∇ equipa a $\mathbb{E}$ con la estructura de un fibrado vectorial holomorfo, que denotamos por E, mientras que la ecuación $\nabla^{0,1}\varphi = 0$ implica que podemos ver φ como un endomorfismo holomorfo «torcido» $\varphi : E \to E \otimes K_X$, donde K_X es el fibrado de línea canónico de X. Estos pares (E, φ) se denominan *fibrados de Higgs*.

Se comprueba fácilmente que los fibrados de Higgs (E, φ) que se obtienen a partir de una solución irreducible a las ecuaciones de Hitchin deben verificar la *condición de estabilidad*, que afirma que todo subfibrado φ-invariante $F \subset E$ debe tener menor grado que E. Recíprocamente, es un teorema de Hitchin [Hit87a] (y de Simpson [Sim88], en mayor generalidad) que todo fibrado de Higgs estable surge de esta forma.

El *espacio de móduli de Hitchin* o *espacio de móduli de fibrados de Higgs* se puede construir tanto con herramientas algebro-geométricas, usando los esquemas *quot* de Grothendieck y la teoría geométrica de invariantes (GIT), como en Nitsure [Nit91] y Simpson [Sim94a, Sim94b], o como cociente kähleriano, como en el artículo de Hitchin [Hit87a]. La equivalencia de ambas construcciones es una consecuencia del teorema de Hitchin y Simpson.

De hecho, el espacio de móduli de Hitchin se puede obtener como un cociente hiperkähleriano y es por tanto una variedad hiperkähleriana. Recordamos que una variedad hiperkähleriana es una variedad riemanniana equipada con tres estructuras complejas I, J y K, que son kählerianas con respecto a la métrica riemanniana y que satisfacen las relaciones cuaterniónicas $I^2 = J^2 = K^2 = IJK = -1$. Cuando el espacio de móduli de Hitchin se construye de esta manera, la estructura I es claramente la que surge naturalmente del hecho de que parametriza fibrados de Higgs. Las otras estructuras complejas surgen de un espacio de móduli distinto, como explicamos a continuación.

La correspondencia de Hodge no abeliana

Si (∇, φ) es una solución a las ecuaciones de Hitchin, podemos considerar el operador

$$D = \nabla + \varphi - \varphi^\dagger,$$

que define una conexión compleja en el fibrado $\mathbb{E}$. Más aún, como asumimos que las ecuaciones de Hitchin se cumplen, la conexión D es plana. Tomando la holonomía, obtenemos una representación r-dimensional del grupo fundamental de X. Recíprocamente, es un teorema de Corlette [Cor88] (y de Donaldson [Don87] en el planteamiento original del artículo de Hitchin [Hit87a]) que los fibrados planos que provienen de representaciones *reductivas* admiten una métrica que es *armónica*, lo que implica que la conexión D proviene de una solución a las ecuaciones de Hitchin.

El espacio de móduli que clasifica las representaciones se conoce como la *variedad de caracteres* y es naturalmente una variedad afín sobre $\mathbb{C}$. Como consecuencia de los resultados de Corlette y Donaldson, la variedad de caracteres es también homeomorfa al espacio de móduli de Hitchin (más aún, estos espacios son difeomorfos en los puntos regulares y, de hecho, son isomorfos como variedades analíticas reales). Las estructuras complejas (isomorfas) J y K pueden entenderse como provenientes de la estructura compleja natural de la variedad de caracteres.

La teoría de fibrados de Higgs, las ecuaciones de Hitchin y los resultados de Corlette, Donaldson, Hitchin y Simpson siguen siendo válidos cuando se consideran G-fibrados principales, para G un grupo complejo reductivo, en vez de fibrados vectoriales (y se recupera el caso de fibrados vectoriales cuando $G = GL_r(\mathbb{C})$).

The Hitchin fibration

Otra característica muy importante de la teoría de fibrados de Higgs es la *fibración de Hitchin*. Si $\mathscr{M}_{\text{Higgs}}(r)$ denota el stack de móduli de los fibrados de Higgs de rango r, la fibración de Hitchin se define como sigue

$$\begin{aligned} h : \mathscr{M}_{\text{Higgs}}(r) &\longrightarrow \mathscr{B} = \bigoplus_{i=1}^{r} H^0(X, K_X^i) \\ (E, \varphi) &\longmapsto (b_1(\varphi), \dots, b_r(\varphi)), \end{aligned}$$

para $b_1(\varphi), \dots, b_r(\varphi)$ los coeficientes del polinomio característico

$$\det(\varphi - T\,\mathrm{id}) = T^r + b_1(\varphi)T^{r-1} + \cdots + b_r(\varphi).$$

Más generalmente, se puede considerar la fibración de Hitchin para el stack de móduli $\mathscr{M}_{\text{Higgs}}(G)$ de G-fibrados de Higgs, para G cualquier grupo reductivo de rango r, tomando como b_i los generadores del anillo de polinomios invariantes $\mathbb{C}[\mathfrak{g}]^G$, para la acción adjunta de G sobre $\mathfrak{g}$, y la *base de Hitchin* $\mathscr{B} = \bigoplus_{i=1}^{r} H^0(X, K_X^{d_i})$, para $d_i = \deg b_i$. Esta aplicación fue introducida por Hitchin en su artículo seminal [Hit87b]. El resultado principal de este artículo es que, para el caso de fibrados vectoriales, la fibra genérica de un elemento $b \in \mathscr{B}$ es una variedad abeliana. Concretamente, es isomorfa al grupo de Picard de una cierta cubierta ramificada $Y_b \to X$ construido a partir de b y llamado la *curva espectral* asociada a b. Hitchin también extiende esto a los grupos clásicos, donde las fibras son isomorfas a unas ciertas *variedades de Prym* asociadas a la curva espectral.

Los resultados de Hitchin se pueden generalizar al caso ligeramente más general de los fibrados de Higgs *torcidos*, definidos en la misma forma que los fibrados de Higgs, pero donde el fibrado canónico se sustituye por cualquier otro fibrado de línea $L \to X$. Esto fue considerado por Beauville, Narasimhan y Ramanan [BNR89]. Sin embargo, una propiedad importante del caso $L = K_X$ es que la dimensión de la base de Hitchin $\mathscr{B}$ es exactamente la mitad de la dimensión del espacio de móduli y por tanto la fibración de Hitchin es un ejemplo de un *sistema algebraicamente completamente integrable*.

Donagi y Gaitsgory [DG02] generalizaron los resultados de Hitchin a cualquier grupo reductivo G, estudiando los *centralizadores regulares* y la estructura de *gerbe* de la fibración de Hitchin. Sus resultados se entienden mejor a la luz del «punto de vista *stacky*» de Ngô [Ng10]. En particular, Ngô usó las propiedades de la fibración de Hitchin para dar una demostración del Lema Fundamental de Langlands–Shelstad para las álgebras de Lie.

Explicamos aquí lo que es este punto de vista «*stacky*» de Ngô, ya que será de gran utilidad en lo que sigue. Se empieza considerando el stack cociente $[\mathfrak{g}/G]$ de $\mathfrak{g}$ por la acción adjunta de G. Por construcción, para cualquier $\mathbb{C}$-esquema S, el grupoide $[\mathfrak{g}/G](S)$ de S-puntos de este stack es el grupoide de los pares (E, φ) con E un G-fibrado principal y φ una sección del fibrado asociado $E(\mathfrak{g})$. Ngô estudia las propiedades locales de la fibración de Hitchin considerando la teoría de invariantes del morfismo natural $[\mathfrak{g}/G] \to \mathfrak{g} /\!\!/ G$ del stack cociente al cociente GIT. En efecto, la fibración de Hitchin puede recuperarse a partir de esto como sigue. La acción natural por homotecia de $\mathbb{C}^*$ en $\mathfrak{g}$ conmuta con la acción adjunta de G, de modo que se pueden considerar los stacks cociente $[\mathfrak{g}/(G\times\mathbb{C}^*)]$ y $[(\mathfrak{g} /\!\!/ G)/\mathbb{C}^*]$. El stack de G-fibrados de Higgs L-torcidos en X puede identificarse con el stack de aplicaciones $X \to [\mathfrak{g}/(G\times\mathbb{C}^*)]$ que cubren la aplicación natural $X \to [\operatorname{Spec}\mathbb{C}/\mathbb{C}^*] = \mathbb{B}\mathbb{C}^*$ definida por L, mientras que la base de Hitchin $\mathscr{B} = \bigoplus_{i=1}^r H^0(X, L^{d_i})$ es el stack de las aplicaciones $X \to [(\mathfrak{g} /\!\!/ G)/\mathbb{C}^*]$ que cubren la misma aplicación definida por L.

El punto de vista *stacky* permite obtener la estructura de gerbe de la fibración de Hitchin a partir de la estructura de gerbe de $[\mathfrak{g}/G] \to \mathfrak{g} /\!\!/ G$. Se hace de la siguiente manera. Primero se considera el *esquema en grupos de centralizadores* $I \to \mathfrak{g}$, con fibras $I_x = \{g \in G : \mathrm{Ad}_g(x) = x\}$. Ngô [Ng10] muestra que, sobre el lugar de puntos regulares $\mathfrak{g}^{reg} \subset \mathfrak{g}$, el esquema en grupos I desciende a un esquema en grupos J sobre $\mathfrak{g} /\!\!/ G$. Este J se llama el *centralizador regular* y se puede probar que $[\mathfrak{g}/G] \to \mathfrak{g} /\!\!/ G$ tiene la estructura de un gerbe atado por J (equivalentemente, de un $\mathbb{B}J$-torsor).

Dualidad de Langlands y simetría mirror

Existe una dualidad para fibraciones de Hitchin con grupos de estructura duales-Langlands, por los resultados de Donagi y Pantev [DP12]. Más precisamente, podemos considerar G y $\check{G}$ dos grupos complejos reductivos duales en el sentido de Langlands (esto es, G y $\check{G}$ tienen datos radicales basados duales), y sus respectivas fibraciones de Hitchin $h_G : \mathscr{M}_{\mathrm{Higgs}}(G) \to \mathscr{B}(G)$ y $h_{\check{G}} : \mathscr{M}_{\mathrm{Higgs}}(\check{G}) \to \mathscr{B}(\check{G})$. Sea $T \subset G$ un toro maximal, que identificamos con $\mathfrak{t}/X_*(T)$, donde $\mathfrak{t}$ denota el álgebra de Lie de T y $X_*(T)$ el retículo de cocaracteres. Podemos considerar $\check{T} \subset \check{G}$ el correspondiente toro maximal, isomorfo al toro dual de G

$$\check{T} \cong T^\vee = \mathfrak{t}^*/X^*(T),$$

donde $X^*(T)$ denota el retículo de caracteres. Si escogemos una forma bilineal invariante en $\mathfrak{g}$, podemos identificar canónicamente $\mathfrak{t}$ y $\mathfrak{t}^*$, y también $X^*(T)$ y $X_*(T)$ y por tanto T y $\check{T}$.

Como explicamos antes, para valores genéricos de $b \in \mathscr{B}(G)$ la fibra de Hitchin $\mathscr{M}_b$ es una variedad abeliana. De hecho, la estructura de gerbe que explicamos antes implica que estas fibras son torsores bajo un cierto stack de Picard $\mathcal{P}_a$. El resultado que obtienen Donagi y Pantev es que la elección de una forma bilineal invariante induce un isomorfismo entre las bases de Hitchin $\mathscr{B}(G) \to \mathscr{B}(\check{G})$, y que, genéricamente, las fibras de Hitchin son duales, con la dualidad dada por una transformada de Fourier–Mukai en los stacks de Picard $\mathcal{P}_a$. Siguiendo el trabajo de Kapustin y Witten [KW07], esto puede interpretarse en términos de Física como el «límite clásico» de la correspondencia de Langlands geométrica.

La dualidad de los sistemas de Hitchin se relaciona con la teoría física de la *simetría* mirror. Esta teoría da una correspondencia bajo la cual ciertos objetos llamados (B, A, A)-branas se asignan a otros llamados (B, B, B)-branas. Estas (B, A, A)-branas son elementos de una cierta categoría de Fukaya, y están soportadas en subvariedades del espacio de móduli de Hitchin que son kählerianas con respecto de la primera estructura compleja I y Lagrangianas con respecto a J y K.

Por otra parte, las (B, B, B)-branas son objetos de una categoría derivada de haces, soportados en subvariedades hiperkählerianas.

LA HISTORIA MULTIPLICATIVA

Monopolos singulares y fibrados de Higgs multiplicativos

La reducción dimensional de las ecuaciones de Yang–Mills auto-duales en $\mathbb{R}^4$ por traslación en una sola dirección da lugar a la *ecuación de Bogomolny* en $\mathbb{R}^3$:

$$F_\nabla = *\nabla\Phi.$$

Esta ecuación está bien definida sobre el producto $Y = S^1 \times X$ de una circunferencia y una superficie de Riemann y, más generalmente, se puede considerar la *ecuación de Hermite–Bogomolny*:

$$F_\nabla - *\nabla\Phi = iC\omega_X.$$

Esta es una ecuación para un par (∇, Φ) formada por una conexión hermítica ∇ en un fibrado vectorial hermítico $\mathbb{E} \to S^1 \times X$ y $\Phi \in \Omega^0(X, \mathfrak{u}_h(\mathbb{E}))$ es una sección del fibrado de endomorfismos que preservan la métrica hermítica. El elemento C es un múltiplo de la identidad en el grupo de automorfismos de $\mathbb{E}$. La misma ecuación se puede considerar para cualquier grupo reductivo complejo si se reemplazan los fibrados vectoriales for G-fibrados principales y, en vez de métricas hermíticas, se consideran reducciones del grupo de estructura de G a un subgrupo compacto maximal $K \subset G$. El elemento C se reemplaza entonces por un elemento central del álgebra de Lie $\mathfrak{k}$ de K.

Estamos interesados en considerar soluciones a la ecuación de Hermite–Bogomolny con *singularidades de tipo Dirac*. Por definición, estas soluciones tienen un conjunto finito de puntos singulares $y_1, \ldots, y_n \in Y$, y en torno a cada uno de estos y_i se puede encontrar una pequeña bola 3-dimensional sobre la cual el grupo de estructura del fibrado $\mathbb{E}$ reduce a un toro maximal y el par (∇, Φ) es aproximado por la imagen del monopolo magnético de Dirac a través de un cierto cocaracter λ_i de G (entendido como un homomorfismo de grupos de Lie compactos $\lambda_i : U(1) \to K$). Las soluciones a la ecuación de Hermite–Bogomolny con singularidades de tipo Dirac se llaman *monopolos singulares*.

La ecuación de Hermite–Bogomolny se puede dividir en tres ecuaciones

$$\begin{cases} F_{\nabla,X} - *\nabla_t\Phi = iC\omega_X, \\ [\nabla_X^{0,1}, \nabla_t - i\Phi dt] = 0, \\ [\nabla_X^{1,0}, \nabla_t + i\Phi dt] = 0. \end{cases}$$

Aquí, hemos separado la conexión en una parte relativa X y una parte relativa a S^1, en la forma $\nabla = \nabla_X + \nabla_t$ y, a su vez, hemos separado la parte relativa a X como $\nabla_X = \nabla_X^{1,0} + \nabla_X^{0,1}$. La primera de estas ecauciones se denomina la *ecuación real*, mientras que las otras dos, que son equivalentes, se llaman las *ecuaciones complejas*.

Los dos operadores $\nabla_X^{0,1}$ y $\nabla_t - i\Phi dt$ definen lo que se conoce como una *estructura mini-holomorfa* en $\mathbb{E}$; éste es un término introducido por Mochizuki

[Moc22]. Por una parte, el operador $\nabla_X^{0,1}$ induce una estructura holomorfa en cada una de las fibras $\mathbb{E}_t = \mathbb{E}|_{\{e^{it}\}\times X}$, y denotamos el fibrado holomorfo resultante por E_t. Por otra parte, tomando el transporte paralelo del operador $\nabla_t - i\Phi dt$ a lo largo de un intervalo $[t_1, t_2]$ se obtiene un homomorfismo

$$\varphi_{t_1,t_2} : \mathbb{E}_{t_1} \longrightarrow \mathbb{E}_{t_2}$$

llamado la *aplicación de* scattering. Las ecuaciones complejas implican que la aplicación de scattering define un homomorfismo holomorfo $E_{t_1} \to E_{t_2}$. Con más precisión, la aplicación de scattering es holomorfa cuando $[t_1, t_2] \times X$ no contiene ningún punto singular, y meromorfa cuando sí lo contiene.

En particular, si tomamos transporte paralelo a lo largo de toda la circunferencia S^1, obtenemos un par (E, φ), con $E = E_0 = E_{2\pi} \to X$ un G-fibrado holomorfo y φ un automorfismo meromorfo. Esto es por definición un*G-fibrado de Higgs multiplicativo*. Si restringimos φ a un disco formal en torno a un punto singular $x_i \in X$ de φ, obtenemos un elemento del grupo de lazos formales $G(F)$, con $F = \mathbb{C}((z))$ el cuerpo de series de Laurent formales en la variable formal z, que está bien definido salvo multiplicación por la izquierda y por la derecha por elementos del espacio de arcos formales $G(\mathcal{O})$, para $\mathcal{O} = \mathbb{C}[[z]]$ el anillo de las series de potencias formales. Es bien sabido que las $G(\mathcal{O}) \times G(\mathcal{O})$-órbitas de $G(F)$ están parametrizadas por los cocaracteres dominantes de G (ver Proposition 1.7.1). Si (E, φ) es un G-fibrado de Higgs multiplicativo obtenido a partir de un monopolo singular, entonces el cocaracter λ_i obtenido de la aplicación de scattering en torno a alguna singularidad x_i es exactamente el mismo cocaracter que aparece en la singularidad de tipo Dirac.

Los fibrados de Higgs multiplicativos que se obtienen a partir de un monopolo singular irreducible deben verificar una cierta *condición de estabilidad* (ver Definition 3.2.12 para más detalles). Recíprocamente, es un teorema de Charbonneau y Hurtubise [CH11], y de Smith [Smi16] en el caso general de grupos reductivos, que todo fibrado de Higgs multiplicativo estable se obtiene de un monopolo singular.

El espacio de móduli de monopolos se puede construir como cociente kähleriano y, cuando X tiene fibrado canónico K_X trivial (lo que sucede tan solo si X tiene género 1), es de hecho un cociente hiperkähleriano. La forma algebraica simpléctica en el espacio de móduli que se obtiene a partir de esta estructura hiperkähleriana fue también construida independientemente desde el punto de vista de los fibrados de Higgs multiplicativos en el artículo de Hurtubise y Markman [HM02].

La fibración de Hitchin multiplicativa

Sea $\boldsymbol{x} = (x_1, \dots, x_n)$ una tupla de puntos en la superficie de Riemann X y sea $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ una tupla de cocaracteres dominantes de G. Denotamos por $\mathscr{M}_{\boldsymbol{x},\boldsymbol{\lambda}}(G)$ el stack de móduli de los G-fibrados de Higgs multiplicativos (E, φ) con una singularidad en cada x_i determinada por el cocaracter λ_i. Consideremos el espacio

$$\mathscr{B}_{x,\lambda}(G) = \bigoplus_{i=1}^{r} H^0(X, \mathscr{O}(\langle \omega_i, \boldsymbol{\lambda} \cdot \boldsymbol{x} \rangle)),$$

donde $\omega_1, \ldots, \omega_r$ son los pesos fundamentales de G y, si $\boldsymbol{\lambda} \cdot \boldsymbol{x} = \sum_{i=1}^{n} \lambda_i x_i$ es un divisor con valores en los cocaracteres dominantes, entonces $\langle \omega_i, \boldsymbol{\lambda} \cdot \boldsymbol{x} \rangle$ es el divisor

$$\langle \omega_i, \boldsymbol{\lambda} \cdot \boldsymbol{x} \rangle = \sum_{j=1}^{n} \langle \omega_i, \lambda_j \rangle x_j.$$

Sea $\rho_i : G \longrightarrow GL(V_i)$ la representación fundamental de G de peso más alto ω_i, y sea $b_i = \mathrm{tr}(\rho_i)$ su traza. Los polinomios $b_1, \ldots, b_r$ generan el anillo de polinomios invariantes $\mathbb{C}[G]^G$ bajo la acción adjunta de G en sí mismo. La *fibración de Hitchin multiplicativa* se define ahora como la aplicación

$$\begin{aligned} \mathscr{M}_{x,\lambda}(G) &\longrightarrow \mathscr{B}_{x,\lambda}(G) \\ (E, \varphi) &\longmapsto (b_1(\varphi), \ldots, b_r(\varphi)). \end{aligned}$$

La fibración de Hitchin multiplicativa fue estudiada originalmente por Hurtubise y Markman [HM02], que en particular probaron que, cuando X tiene género 1, define un sistema algebraicamente completamente integrable.

El punto de vista de los monoides

Un punto de vista muy apropiado para el estudio de los fibrados de Higgs multiplicativos es considerarlos en el contexto de la teoría de *monoides reductivos*. Un monoide reductivo es un monoide algebraico M sobre $\mathbb{C}$ (esto es, un objeto en monoides en la categoría de $\mathbb{C}$-esquemas) tal que su grupo de unidades $M^\times$ es reductivo. El ejemplo prototípico es el monoide Mat_n de las matrices $n \times n$, que tiene como grupo de unidades el grupo general lineal, $\mathrm{Mat}_n^\times = GL_n$. El punto de vista de los monoides permite ver (cuando G es semisimple) la fibración de Hitchin multiplicativa en un marco *stacky* muy similar al punto de vista *stacky* de Ngô para la fibración de Hitchin.

Supongamos que G es semisimple y simplemenete conexo y sea M un monoide tal que that el grupo derivado de $M^\times$ es igual a G. El cociente GIT $\mathbb{A}_M = M /\!\!/ (G \times G)$ es una Z-variedad tórica, con $Z = Z^0_{M^\times}$ la componente conexa neutra del centro del grupo de unidades $M^\times$, llamada la *abelianización* del monoide M. Un monoide es *muy plano* si la aplicación cociente $M \to \mathbb{A}_M$ es plana, dominante y con fibras íntegras. Un morfismo de monoides $M_1 \to M_2$ es *excelente* si el cuadrado conmutativo inducido con las abelianizaciones es Cartesiano (en otras palabras, si $M_1 = M_2 \times_{\mathbb{A}_{M_2}} \mathbb{A}_{M_1}$). La categoría de los monoides muy planos con parte semisimple G y con morfismos excelentes tiene un objeto universal llamado el monoide de Vinberg o el monoide envolvente de G, y denotado por Env(G). El grupo unidad de Env(G) es el grupo $G_+ = (G \times T)/Z$, para $T \subset G$ a maximal torus, de modo que su centro es $Z_+ \cong T^{ad}$, y su abelianización es la T^{ad}-variedad tórica $\mathbb{A}_{\mathrm{Env}(G)}$ determinada por el semigrupo de pesos

$$P_+(\mathbb{A}_{\mathrm{Env}(G)}) = \mathbb{Z}_+ \{\alpha_1, \ldots, \alpha_r\},$$

para $\alpha_1, \ldots, \alpha_r$ las raíces simples de T.

Nótese que los S-puntos del stack cociente $[\mathbb{A}_{\mathrm{Env}(G)}/T^{\mathrm{ad}}]$ consisten en pares (L, s) formados por un T^{ad}-torsor L sobre S y una sección s del fibrado asociado $L(\mathbb{A}_{\mathrm{Env}(G)})$. Un par $(\mathbf{x}, \boldsymbol{\lambda})$ formado por una tupla de n puntos de X y una tupla de n cocaracteres dominantes define naturalmente un X-punto de $[\mathbb{A}_{\mathrm{Env}(G)}/T^{\mathrm{ad}}]$, tomando L como el T^{ad}-fibrado definido tomando cada λ_i como función de transición en un disco en torno a x_i y s la sección natural, que no se anula, del fibrado

$$L(\mathbb{A}_{\mathrm{Env}(G)}) = \bigoplus_{i=1}^{r} \mathcal{O}\left(\sum_{j=1}^{n} \langle \alpha_i, \lambda_j \rangle x_i\right).$$

En efecto, esta sección que no se anula existe como consecuencia de que los caracteres λ_j son dominantes. El stack de móduli de los G-fibrados de Higgs multiplicativos se recupera entonces como el stack de las aplicaciones $X \longrightarrow [\mathrm{Env}(G)/(G \times T^{\mathrm{ad}})]$ que cubren la aplicación natural $X \longrightarrow [\mathbb{A}_{\mathrm{Env}(G)}/T^{\mathrm{ad}}]$ inducida por el par $(\mathbf{x}, \boldsymbol{\lambda})$.

INVOLUCIONES

Involuciones, variedades simétricas y formas reales

Esta memoria está principalmente dedicada a la interacción entre los fibrados de Higgs multiplicativos y los monopolos con las involuciones holomorfas θ del grupo de estructura G. Una *involución* de G es un automorfismo $\theta \in \mathrm{Aut}(G)$ de orden 2 (esto es, $\theta^2 = \mathrm{id}$). Podemos considerar el subgrupo de puntos fijos asociado a una involución

$$G^{\theta} = \{g \in G : \theta(g) = g\}.$$

Además, la involución θ induce la *descomposición de Cartan* del álgebra de Lie $\mathfrak{g}$ de G en la forma

$$\mathfrak{g} = \mathfrak{g}^{\theta} \oplus \mathfrak{m}^{\theta},$$

donde $\mathfrak{g}^{\theta}$ y $\mathfrak{m}^{\theta}$ son los espacios propios con autovalores $+1$ y -1 de θ, respectivamente. El espacio $\mathfrak{g}^{\theta}$ es en efecto el álgebra de Lie de G^{θ}, mientras que $\mathfrak{m}^{\theta}$ es el espacio tangente a la identidad en la variedad simétrica G/G^{θ}.

En general, un *subgrupo simétrico asociado a* θ es un subgrupo cerrado $H \subset G$ tal que $G_0^{\theta} \subset H \subset G_{\theta}$, donde $G_0^{\theta} = (G^{\theta})^0$ es la componente conexa neutra de G^{θ} y $G_{\theta} = \left\{g \in G : g\theta(g)^{-1} \in Z_G\right\}$ es igual al normalizador de G^{θ}. Aquí, Z_G denota el *centro* de G. Un espacio homogéneo G/H, donde $H \subset G$ es un subgrupo simétrico, se llama una *variedad simétrica*.

La variedad simétrica G/G^{θ} puede verse dentro del grupo G como la subvariedad $M^{\theta} = \left\{g\theta(g)^{-1} : g \in G\right\}$. Más generalmente, podemos considerar la acción de conjugación θ-torcida $g *_{\theta} x = gx\theta(g)^{-1}$ de G en sí mismo. Las órbitas θ-torcidas son espacios homogéneos de la forma G/G^{θ_s}, con $\theta_s = \mathrm{Int}_s \circ \theta$, y Int_s el automorfismo interno dado por conjugación por s. El automorfismo θ_s será una involución sólo si s pertenece al subconjunto

$$S_{\theta} = \{s \in G : s\theta(s) \in Z_G\}.$$

Esto permite identificar el subconjunt S_θ con los elementos de la clase $a \in \mathrm{Out}_2(G)$ de θ en el grupo $\mathrm{Out}_2(G) = \mathrm{Aut}_2(G)/\mathrm{Int}(G)$.

Además, los grupos G^θ y $G^{\theta'}$ (y por tanto las variedades simétricas correspondientes) se identifican si θ y θ' son conjugadas por un automorfismo interno. Esto induce una relación de equivalencia $\sim$ en el grupo de involuciones $\mathrm{Aut}_2(G)$, y la restricción de la aplicación natural $\mathrm{Aut}_2(G) \to \mathrm{Out}_2(G)$ a $\mathrm{Aut}_2(G)/\sim$ es la *aplicación* clique $\mathrm{cl} : \mathrm{Aut}_2(G)/\sim \longrightarrow \mathrm{Out}_2(G)$ de García-Prada y Ramanan [GPR19]. La fibra de un elemento $a \in \mathrm{Out}_2(G)$ por la aplicación clique se puede identificar con el cociente

$$\mathrm{cl}^{-1}(a) = S_\theta/(G \times Z_G)$$

de S_θ (para cualquier representante θ de a) por la acción de $G \times Z_G$ definida como

$$(g, z) * x = zg *_\theta x.$$

Es un resultado clásico de Cartan [Car14] que las involuciones y las formas reales de G se pueden identificar. En efecto, para cada involución θ existe una forma real compacta maximal K cuya conjugación de G asociada σ_K conmuta con θ, y se puede considerar la forma real $\sigma_\theta = \sigma_K \circ \theta$. Recíprocamente, dada cualquier conjugación σ existe una involución θ_σ tal que $\sigma \circ \theta_\sigma$ es la conjugación de una forma real compacta maximal.

Fibrados de Higgs para formas reales

Sea $\theta \in \mathrm{Aut}_2(G)$ una involución de G. Un (G, θ)-*fibrado de Higgs* es un par (E, φ), donde $E \to X$ es un G^θ-fibrado principal y φ es una sección del fibrado $E(\mathfrak{m}^\theta) \otimes K_X$. Aquí, $E(\mathfrak{m}^\theta)$ es el fibrado asociado a la acción natural de G^θ en $\mathfrak{m}^\theta$ inducida por la acción adjunta de G en $\mathfrak{g}$.

En la literatura, los (G, θ)-fibrados de Higgs suelen recibir el nombre de $G_\mathbb{R}$-*fibrados de Higgs*, donde $G_\mathbb{R}$ es la forma real de G asociada a θ. La razón detrás de esta nomenclatura es que bajo la correspondencia de Hodge no abeliana, estos objetos dan lugar a representaciones del grupo fundamental de X que factorizan a través del grupo real $G_\mathbb{R}$ (ver [GP20] para más referencias).

En el trabajo de García-Prada y Ramanan [GPR19], se demuestra que los (G, θ)-fibrados de Higgs forman el lugar de puntos fijos de la involución natural en el espacio de móduli de G-fibrados de Higgs dada por

$$(E, \varphi) \mapsto (\theta(E), -\theta(\varphi)).$$

La involución ι_-^θ que estudiamos en esta memoria es una «versión multiplicativa» de ésta. Una consecuencia sencilla de este hecho es que los (G, θ)-fibrados de Higgs forman una subvariedad del espacio de móduli de Hitchin, que es kähleriana con respecto a la primera estructura compleja I y lagrangiana con respecto a las otras dos.

Se ha sugerido que los fibrados de Higgs para formas reales $G_\mathbb{R}$ de G definen (B, A, A)-branas en el espacio de móduli de G-fibrados de Higgs correspondientes a $\check{G}_{G_\mathbb{R}} \subset \check{G}$, el grupo dual de la forma real $G_\mathbb{R}$ introducido por Nadler [Nad05]. Esta conjetura surge de una descripción en el lenguaje de teoría *gauge* de la construcción del grupo de Nadler, dada por Gaiotto y Witten [GW09]. Ver la Sección 1.4 para más información sobre el grupo dual.

La teoría de invariantes de la acción de G^θ en $\mathfrak{m}^\theta$ fue estudiada por Kostant y Rallis [KR71]. Su trabajo da una generalización del isomorfismo de Chevalley, probando que el cociente GIT $\mathfrak{m}^\theta /\!\!/ G^\theta$ es isomorfo a $\mathfrak{a}/W_\theta$, donde $\mathfrak{a} \subset \mathfrak{m}^\theta$ es una subálgebra abeliana maximal y W_θ es el *pequeño grupo de Weyl* asociado a la involución θ (ver Proposition 1.3.11). Esto permite definir una versión de la fibración de Hitchin para (G,θ)-fibrados de Higgs, también conocida como la fibración de Hitchin para grupos reales o la fibración de Hitchin para pares simétricos. Esta versión de la fibración de Hitchin fue introducida en la tesis de Peón-Nieto [PN13], y explorada posteriormente en su trabajo con García-Prada and Ramanan [GPPNR18, GPPN23]. Una descripción completa de la fibración de Hitchin para pares simétricos aparecerá en un trabajo próximo de Morrissey y Hameister [HM] (el lector interesado puede consultar la charla de Hameister [Ham23]).

Involuciones en el espacio de fibrados de Higgs multiplicativos

Una involución $\theta \in \mathrm{Aut}_2(G)$ induce involuciones en el espacio de fibrados de Higgs multiplicativos de una forma muy similar a como lo hace en el espacio de fibrados de Higgs. Concretamente, para cualquier G-fibrado de Higgs (E,φ) podemos considerar

$$\iota_\pm^\theta(E,\varphi) = (\theta(E), \theta(\varphi)^{\pm 1}).$$

Al nivel de las clases de isomorfía, las involuciones $\iota_\pm^\theta$ sólo dependen de la clase $a \in \mathrm{Out}_2(G)$ de θ en $\mathrm{Out}_2(G)$, y por tanto tiene sentido denotar $\iota_\pm^a$. Al nivel de los fibrados mini-holomorfos en $Y = S^1 \times X$, se puede mostrar que estas involuciones corresponden a

$$\iota_\pm^\theta(\mathcal{E}) = \zeta_\pm^* \theta(\mathcal{E})$$

donde $\mathcal{E} \to Y$ denota el fibrado mini-holomorfo correspondiente a (E,φ), y $\zeta_\pm : Y \to Y$ son las involuciones

$$\begin{aligned} \zeta_\pm : S^1 \times X &\longrightarrow S^1 \times X \\ (e^{it}, x) &\longmapsto (e^{\pm it}, x). \end{aligned}$$

Los puntos fijos de la involución ι_-^a incluyen a los (G,θ)-*fibrados de Higgs multiplicativos*, que recordamos que son pares (E,φ) con $E \to X$ un G^θ-fibrado principal holomorfo y φ una sección meromorfa del fibrado de variedades simétricas asociado $E(G/G^\theta)$. Esto tiene sentido porque los (G,θ)-fibrados de Higgs multiplicativos se extienden naturalmente a G-fibrados de Higgs multiplicativos, ya que la acción de multiplicación por la izquierda de G en G/G^θ se corresponde a la acción de conjugación θ-torcida en M^θ y por tanto G^θ actúa en M^θ por conjugación.

Las singularidades locales de los (G,θ)-fibrados de Higgs multiplicativos pueden fijarse de una forma similar a como se hace para los G-fibrados de Higgs multiplicativos. Si restringimos la sección φ a un disco formal en torno a un punto singular $x_i \in X$, obtenemos un elemento del espacio de lazos formales

$(G/G^\theta)(F)$, bien definido salvo multiplicación por elementos de $G(\mathcal{O})$. Las $G(\mathcal{O})$-órbitas de $(G/G^\theta)(F)$ están parametrizadas por cocaracteres antidominantes del toro $A_{G^\theta} = A/(A \cap G^\theta)$, donde A es un *toro θ-escindido maximal* (esto es, A es maximal entre todos los toros de G tales que $\theta(a) = a^{-1}$ para todo $a \in A$).

A parte de los (G, θ)-fibrados de Higgs multiplicativos, aparecen otros objetos como puntos fijos de $\iota^a_\pm$. Esto es una diferencia esencial entre los casos «aditivo» y el multiplicativo, que surge del hecho de que, aunque $\mathfrak{m}^\theta$ es precisamente el conjunto de los puntos anti-fijos de $\mathfrak{g}$ por la involución inducida por θ, el conjunto $S^\theta = \left\{s \in G : \theta(s) = s^{-1}\right\}$ de puntos anti-fijos contiene a la variedad simétrica $M^\theta \cong G/G^\theta$, pero también otras componentes, isomorfas a las variedades simétricas G/G^{θ_s}, para $s \in S^\theta$ y $\theta_s = s\theta(-)s^{-1}$. Los otros objetos son por tanto pares (E, φ) con E un G^θ-fibrado y φ una sección meromorfa del fibrado $E(G/G^{\theta_s})$ con G^θ actuando en G/G^{θ_s} por multiplicación por la izquierda. Los valores posibles de s que pueden aparecer en los puntos fijos están restringidos por el valor de los cocaracteres en las singularidades (ver Corollary 2.3.9).

Como en el caso de los (G, θ)-fibrados de Higgs, se puede construir una aplicación de Hitchin para los (G, θ)-fibrados de Higgs multiplicativos. La construcción se basa en el estudio de Richardson [Ric82b] de la teoría de invariantes de la acción de G^θ en M^θ. Richardson prueba que el cociente GIT $M^\theta /\!\!/ G^\theta$ es isomorfo a A/W_θ, para A un toro θ-escindido maximal, y W_θ el pequeño grupo de Weyl asociado a θ.

El punto de vista de las inmersiones simétricas

El «punto de vista de los monoides» para los G-fibrados de Higgs multiplicativos se puede generalizar a la teoría de los (G, θ)-fibrados de Higgs multiplicativos si se consideran *inmersiones equivariantes de variedades simétricas*. Para acortar, denominamos este tipo de inmersiones simplemente *inmersiones simétricas*.

Empezamos tomando G que sea semisimple y simplemente conexo, y considerando la variedad simétrica G/G^θ. Supongamos que Σ es una inmersión equivariante de una variedad simétrica O_Σ tal que su *parte semisimple* es igual a G/G^θ. Por definición, la parte semisimple de una variedad simétrica $O_\Sigma = G_1/H_1$ es el espacio homogéneo $G_1'/(H_1 \cap G_1')$, donde G_1' es el grupo derivado de G_1. Podemos definir la *abelianización* de la inmersión simétrica Σ como el cociente GIT $\mathbb{A}_\Sigma = \Sigma /\!\!/ G$, que es una A_Σ-variedad tórica, para $A_\Sigma = O_\Sigma/G$. Igual que con los monoides, decimos que una inmersión simétrica Σ es *muy plana* si el cociente $\Sigma \to \mathbb{A}_\Sigma$ es plano, dominante y con fibras íntegras. Un morfismo de inmersiones simétricas $\Sigma_1 \to \Sigma_2$ es *excelente* si $\Sigma_1 = \Sigma_2 \times_{\mathbb{A}_{\Sigma_2}} \mathbb{A}_{\Sigma_1}$. La categoría de las inmersiones simétricas muy planas con parte semisimple G/G^θ y con morfismos excelentes tiene un objeto universal llamado la *inmersión de Guay* o la *inmersión envolvente* de G/G^θ, y denotado por $\mathrm{Env}(G/G^\theta)$. El toro $A_{\mathrm{Env}(G/G^\theta)}$ es igual a $A_{G_\theta} := A/(A \cap G_\theta)$, para $A \subset G$ un toro θ-escindido maximal, y la abelianización $\mathbb{A}_{\mathrm{Env}(G/G^\theta)}$ es la A_{G^θ}-variedad tórica determinada por el semigrupo de pesos

$$P_+(\mathbb{A}_{\mathrm{Env}(G/G^\theta)}) = -\mathbb{Z}_+ \{\bar{\alpha}_1, \ldots, \bar{\alpha}_l\},$$

para $\bar{\alpha}_1, \ldots, \bar{\alpha}_l$ las raíces restringidas simples asociadas a θ (ver Sección 1.3).

Ahora, los S-puntos del stack cociente $[\mathbb{A}_{\mathrm{Env}(G/G^\theta)}/A_{G_\theta}]$ consisten en pares (L, s) formados por un A_{G_θ}-torsor L sobre S y una sección s del fibrado asociado $L(\mathbb{A}_{\mathrm{Env}(G/G^\theta)})$. Un par (x, λ) formado por una tupla de n puntos de X y n cocaracteres antidominantes de A_{G^θ} naturalmente define un X-punto de $[\mathbb{A}_{\mathrm{Env}(G/G^\theta)}/A_{G_\theta}]$, tomando L como el A_{G_θ}-fibrado definido por las funciones de transición dadas por las λ_i y s la sección natural que no se anula de

$$L(\mathbb{A}_{\mathrm{Env}(G/G^\theta)}) = \bigoplus_{i=1}^{l} \mathcal{O}\left(\sum_{j=1}^{n} \langle -\bar{\alpha}_i, \lambda_j \rangle x_i\right),$$

que existe como consecuencia de que los λ_j son antidominantes. El stack de móduli de los (G, θ)-fibrados de Higgs multiplicativos se recupera entonces como el stack de las aplicaciones $X \to [\mathrm{Env}(G/G^\theta)/(G^\theta \times A_{G_\theta})]$ que cubren la aplicación natural $X \to [\mathbb{A}_{\mathrm{Env}(G/G^\theta)}/A_{G_\theta}]$ inducida por el par (x, λ).

SÍNTESIS Y RESULTADOS PRINCIPALES

Esta memoria consiste en tres capítulos, más esta introducción y una discusión final sobre posibles direcciones futuras a seguir basadas en los resulatdos de esta tesis.

El Capítulo 1 cubre la teoría de las variedades simétricas y sus inmersiones. El propósito de este capítulo es esencialmente el de dar algunas nociones preliminares necesarias para entender los siguientes capítulos, especialmente el Capítulo 2. En cualquier caso, el capítulo también contiene algunos resultados originales relativos a la teoría de invariantes de las inmersiones simétricas y la parametrización de sus lazos formales. Comenzamos revisando algunos hechos generales sobre grupos reductivos con involución y explicamos los puntos de vista relacionados relativos a las formas reales y a la cohomología de grupos no abeliana. Tras eso, introducimos las nociones de toro θ-escindido y de subgrupo parabólico θ-escindido y repasamos la construcción del sistema de raíces restringidas y sus retículos de raíces y de pesos asociados. Continuamos revisando la teoría de las inmersiones equivariantes de las variedades simétricas, originalmente desarrollada por Vust [Vus90] como un caso particular de la teoría más general de las variedades esféricas de Luna y Vust [LV83]. Repasamos algunas nociones básicas sobre variedades esféricas afines y describimos el retículo de pesos y el semigrupo de pesos de una variedad simétrica. Recordamos también la definición y las propiedades principales del grupo dual de una variedad simétrica. En la Tabla 1.2 resumimos alguna información sobre las involuciones de los grupos simples.

El capítulo 1 continúa explicando la construcción y las propiedades principales de la *compactificación magnífica* de una variedad simétrica, siendo de particular interés el teorema de estructura local de De Concini y Procesi. Continuamos explicando la teoría de Guay de las inmersiones muy planas de variedades simétricas [Gua01]. En particular, repasamos la noción de la abelianización de una inmersión, la caracterización de Guay de las inmersiones muy planas, y la construcción y propiedades principales de la inmersión envolvente de Guay. Terminamos la sección demostrando un resultado original, la Proposition 1.6.15, relativa a la teoría de invariantes de una inmersión simétrica. En la sección siguiente, estudiamos la parametrización de los lazos formales de una variedad simétrica y sus inmersiones, lo que quiere decir que estudiamos sus espacios de lazos formales y describimos las órbitas del grupo de arcos formales en ellos. En particular, en la Proposition 1.7.7, generalizamos la Proposition 1.8.6, que es un resultado de J. Chi [Chi22]. Concluimos el capítulo revisando la teoría de monoides reductivos como un caso particular de la teoría de Guay de las inmersiones simétricas.

El Capítulo 2 se centra en los fibrados de Higgs multiplicativos sobre la superficie de Riemann compacta X. Empezamos con una sección preliminar revisando la teoría de fibrados de Higgs multiplicativos, tal y como fueron introducidos originalmente por Hurtubise y Markman [HM02], por una parte, y por otra parte tal y como se desarrollaron independientemente en los trabajos de Bouthier, J. Chi and G. Wang [Bou15, BC18, Bou17, Chi22, Wan23] usando el punto de vista de los monoides reductivos. En particular, comenzamos explicando un punto de vista muy general relativo a la aplicación de Hitchin y sus análogos y generalizaciones sugerido por Morrissey y Ngô [MN] (el lector interesado puede consultar la charla de Ngô [Ng23]), para posteriormente explicar dos puntos de vista equivalentes sobre la fibración de Hitchin multiplicativa. También recordamos los argumentos de Hurtubise y Markman a favor de la existencia de un espacio de móduli de fibrados de Higgs multiplicativos simples, y su descripción del espacio tangente y, cuando X tiene género 1, de la forma simpléctica holomorfa en el espacio de móduli.

Continuamos el Capítulo 2 con dos secciones de resultados originales relativos a nuestra generalización de la teoría de fibrados de Higgs multiplicativos de los grupos reductivos a las variedades simétricas. En la Sección 2.2 definimos los (G, θ)-fibrados de Higgs multiplicativos y la aplicación de Hitchin asociada a ellos, y mostramos que es equivalente a una aplicación de Hitchin natural que surge de la teoría de las inmersiones simétricas. Este es el contenido de nuestro Theorem 2.2.5 que es una generalización del resultado de Bouthier, Chi y Wang (nuestro Theorem 2.1.9). También mostramos en la Sección 2.2 cómo los (G, θ)-fibrados de Higgs multiplicativos pueden verse dentro del espacio de móduli de los G-fibrados de Higgs multiplicativos. La última sección del Capítulo 2, la Sección 2.3 da una descripción completa de las involuciones $\iota_{\pm}^{\theta} : (E, \varphi) \mapsto (\theta(E), \theta(\varphi)^{\pm 1})$ en el espacio de móduli de G-fibrados de Higgs multiplicativos, y de sus puntos fijos. Los resultados principales de esta parte son el Theorem 2.3.4 y el Corollary 2.3.9. Concluimos describiendo la interacción de las $\iota_{\pm}^{\theta}$ con la forma simpléctica de Hurtubise–Markman en el Theorem 2.3.10.

El Capítulo 3 estudia la contrapartida del problema estudiado en el anterior Capítulo 2 desde el punto de vista de los fibrados mini-holomorfos y los monopolos. Comenzamos con una sección que introduce el concepto de un G-*fibrado principal mini-holomorfo* sobre $Y = S^1 \times X$ el producto de una circunferencia y una superficie de Riemann compacta. Este concepto es una generalización de la noción de fibrado vectorial mini-holomorfo introducida por Mochizuki [Moc22]. También introducimos las nociones asociadas del par de Chern, singularidades de tipo Dirac y la aplicación de scattering, y explicamos la equivalencia entre los fibrados mini-holomorfos y los fibrados de Higgs multiplicativos. Los fibrados principales mini-holomorfos ya fueron considerados en el trabajo de Smith [Smi16] (él los llama «estructuras holomorfas»). Sin embargo, aquí tratamos el tema de una forma diferente, más intrínseca (ya que no necesitamos emplear representaciones ni derivadas covariantes en fibrados vectoriales asociados) que, por lo que sabemos, aún no ha sido presentada en la literatura.

Las secciones siguientes del Capítulo 3 están dedicadas al estudio de los monopolos. En primer lugar, introducimos la ecuación de Hermite–Bogomolny y la noción de monopolo singular, y repasamos los resultados de Charbonneau–Hurtubise [CH11] y Smith [Smi16], que relacionan los monopolos con los fibrados mini-holomorfos poliestables. Explicamos también la construcción del espacio de móduli de monopolos como un cociente kähleriano, y como un cociente hiperkähleriano cuando X tiene género 1. Finalmente, en la Sección 2.3.9, describimos las involuciones $\iota^{\theta}_{\pm}$ y sus puntos fijos desde el punto de vista de los fibrados mini-holomorfos, y recuperamos los resultados de la Sección 2.3 en la Proposition 3.4.3 y en la Proposition 3.4.4. La sección termina con la descripción de las involuciones en términos de monopolos singulares, y describiendo la interacción de las involuciones con la estructura hiperkähleriana del espacio de móduli de monopolos, cuando X tiene género 1.

La memoria concluye con una exploración de diferentes direcciones futuras de investigación que podrían ramificarse a partir de esta tesis. Comenzamos con varias conjeturas (las Conjectures 1, 2 y 3) relativas a posibles análogos de los resultados de Donagi–Pantev sobre la dualidad de los sistemas de Hitchin para la fibración de Hitchin multiplicativa. La demostración de estas conjeturas es el contenido de trabajo próximo conjunto con Benedict Morrissey [GM]. Continuamos sugiriendo algunos problemas y preguntas relativos a la descripción de la aplicación de Hitchin para (G, θ)-fibrados de Higgs a través del estudio de los *cocientes regulares*, en el sentido de Morrissey y Ngô [MN, Ng23] de la variedad simétrica G/G^{θ} y de la inmersión envolvente $\mathrm{Env}(G/G^{\theta})$ por la acción de G^{θ}. En particular, planteamos la cuestión de la existencia de una sección transversal del cociente GIT $G/G^{\theta} \to (G/G^{\theta}) /\!\!/ G^{\theta}$ que generalice la sección de Kostant–Rallis section [KR71] y la sección transversal de Steinberg [Ste65]. En las Questions 2 y 3 proponemos el estudio de una posible generalización de la teoría de Guay de las inmersiones muy planas de variedades simétricas al contexto más general de las variedades esféricas. En el Problem 2, sugerimos la existencia de una descripción distitna de los fibrados de Higgs multiplicativos en términos de teoría *gauge* que, a diferencia de la teoría de monopolos singulares, está definida intrínsecamente en la superficie de Riemann X y no necesita utilizar el producto con la circunferencia S^1, al menos explícitamente. Esperamos que esto pueda hacerse mediante el estudio de las estructuras kählerianas en los monoides muy planos y sus correspondientes aplicaciones momento para la acción del subgroupo compacto maximal K de G, y aplicando la teoría de pares desarrollada por Mundet i Riera [MiR00]. Terminamos el apartado de direcciones futuras proponiendo el estudio de un «lado de de Rham» en el espacio de móduli de monopolos singulares. Otra dirección interesante posible, que no incluimos en esta sección ya que la dejamos completamente inexplorada, es la posible aplicación de los resultados de esta tesis a una versión «relativa» (en el sentido de Ben-Zvi–Sakellaridis–Venkatesh [BZSV23]) del Lema

Fundamental de Langlands–Shelstad para variedades simétricas, generalizando la tesis de G. Wang [Wan23].

Por comodidad, en los Capítulos 1 y 2 trabajamos en el contexto algebraico en vez de en el contexto holomorfo (que es equivalente). Esto es, consideramos que G es un grupo algebraico reductivo sobre $\mathbb{C}$ y que X es una curva compleja proyectiva lisa, en vez de considerar un grupo reductivo complejo y una superficie de Riemann compacta, respectivamente, y consideramos fibrados principales algebraicos (en vez de holomorfos) sobre X. La equivalencia de ambos contextos es una consecuencia del GAGA de Serre [Ser56]. Se sigue del hecho de que todos los argumentos de estos capítulos son algebraicos que todo lo que hay en ellos (a excepción de los comentarios relativos a formas reales) sigue siendo cierto si sustituimos $\mathbb{C}$ por cualquier cuerpo algebraicamente cerrado de característica 0.

1 THE THEORY OF SYMMETRIC VARIETIES

1.1 REDUCTIVE GROUPS AND HOMOGENEOUS SPACES

Linear algebraic groups and the Jordan decomposition

A *group scheme* over $\mathbb{C}$ is a group object in the category of schemes over $\mathbb{C}$. A *linear algebraic group* over $\mathbb{C}$ is a smooth and affine group scheme over $\mathbb{C}$ of finite type.

The paradigmatical example of a linear algebraic group is the general linear group GL_n, whose closed points are the complex invertible matrices. In fact, every linear algebraic group G admits a closed embedding $\iota : G \hookrightarrow GL_n$. A very useful consequence of this is that linear algebraic groups admit a *Jordan decomposition*.

The Jordan decomposition in linear algebra states any complex matrix M can be decomposed as $M = D+N$, for D a diagonalizable matrix and N a nilpotent matrix. At the level of $GL_n(\mathbb{C})$, this implies that any invertible matrix A decomposes as $A = DU$, for D an invertible diagonalizable matrix and U an unipotent matrix (meaning that $U - I_n$ is nilpotent). This can be extended to linear algebraic groups. An element $g \in G$ is *semisimple* or *diagonalizable* if $\iota(g)$ is diagonalizable, and *unipotent* if $\iota(g)$ is unipotent. This notion can be shown to not depend on the choice of the embedding, and any element $g \in G$ decomposes as $g = g^s g^u$, for g^s semisimple and g^u unipotent. Some modern references covering the basics about linear algebraic groups are Conrad's lecture notes [Con20], and the books of Milne [Mil17] and Humphreys [Hum75].

Algebraic homogeneous spaces

Let G be a linear algebraic group over $\mathbb{C}$ and Σ be a complex algebraic variety. We say that Σ is a (left) G-*variety* if it is endowed with a (left) action of G, that is, with a morphism $\rho : G \times \Sigma \to \Sigma$ such that $\rho(1, x) = x$, for all $x \in \Sigma$, and $\rho(g_1, \rho(g_2, x)) = \rho(g_1 g_2, x)$. We usually write $g \cdot x$ for $\rho(g, x)$. Similarly, one can define right G-varieties and right actions.

A *geometric quotient* Σ/G of a G-variety Σ is a variety defined as the orbit space Σ/G equipped with the quotient topology and the structure sheaf $\mathcal{O}_{\Sigma /\!\!/ G}$ defined as the direct image of the sheaf $\mathcal{O}_\Sigma^G$ of G-invariant regular functions on Σ. Geometric quotients do not exist in general, but they do in some cases. In particular, if Σ is an affine variety and $\mathbb{C}[\Sigma]$ is its ring of regular functions, then, if G is *reductive* (defined below), we can put $\Sigma /\!\!/ G = \operatorname{Spec}(\mathbb{C}[\Sigma]^G)$, which is again an affine variety, called the *GIT quotient* of Σ by G.

Let G be a linear algebraic group and $H \subset G$ a closed subgroup, which acts on H by right translations. In this case the geometric quotient G/H is always known to exist, and it is a quasiprojective variety (see [Tim11, Theorem 1.3]). An (algebraic) *homogeneous space* O is a G-variety such that the action of G on O is transitive. If $x \in O$ is any point, we can consider the orbit map $G \to O$ sending $g \in G$ to $g \cdot x$. If we take $H = G_x$ to be the stabilizer of x, then the orbit map $G \to O$ factors through a map $G/H \to O$. If the orbit map is separable (which we always assume), then the map $G/H \to O$ is an isomorphism. If the group H is reductive, then $G/H = \operatorname{Spec}(\mathbb{C}[G]^H)$ is affine.

Algebraic tori, characters and cocharacters

A *complex algebraic torus* of *rank* r is a linear algebraic group T over $\mathbb{C}$ isomorphic to $(\mathbb{C}^*)^r$. A *character* of a complex algebraic torus T is a homomorphism of algebraic groups $\chi : T \to \mathbb{C}^*$, whereas a *cocharacter* is a homomorphism $\lambda : \mathbb{C}^* \to T$. We denote by $X^*(T)$ and $X_*(T)$ the sets of characters and cocharacters of T, respectively. These are the *character lattice* and the *cocharacter lattice*. Indeed, it is clear that both $X^*(T)$ and $X_*(T)$ are free abelian groups of rank r. Moreover, the two lattices $X^*(T)$ and $X_*(T)$ are in duality, identified by the perfect pairing

$$\begin{aligned} X^*(T) \times X_*(T) &\longrightarrow \mathbb{Z} \\ (\chi, \lambda) &\longmapsto \langle \chi, \lambda \rangle, \end{aligned}$$

for $\langle \chi, \lambda \rangle$ such that

$$(\chi \circ \lambda)(z) = z^{\langle \chi, \lambda \rangle},$$

for every $z \in \mathbb{C}^*$.

The Lie algebra $\mathfrak{t}$ of a complex algebraic torus T of rank r is isomorphic to $\mathbb{C}^r$. The differential of a character $T \to \mathbb{C}^*$ defines an element of $\mathfrak{t}^*$, so we can regard $X^*(T)$ as a sublattice of $\mathfrak{t}^*$. On the other hand, the differential of a cocharacter $\mathbb{C}^* \to T$ can be seen as an element of $\mathfrak{t}$, and we can regard $X_*(T)$ as a sublattice of $\mathfrak{t}$. The closed points of the torus T can be identified with $\mathfrak{t}/X_*(T)$, whereas the quotient $\mathfrak{t}^*/X^*(T)$ defines the *dual torus* $T^\vee$ of T.

Along this document we use the *additive notation* for characters and cocharacters, by regarding elements $\chi \in X^*(T)$ and $\lambda \in X_*(T)$ as elements of $\mathfrak{t}^*$ and $\mathfrak{t}$, respectively, and denoting the corresponding maps $T \to \mathbb{C}^*$ and $\mathbb{C}^* \to T$ as $t \mapsto t^\chi$ and $z \mapsto z^\lambda$. This notation makes sense if we pick isomorphisms of $\mathfrak{t}$ and $\mathfrak{t}^*$ with $\mathbb{C}^r$ and of $X^*(T)$ and $X_*(T)$ with $\mathbb{Z}^r$. In that case, if we let $\chi = (\chi_1, \dots, \chi_r)$ and $\lambda = (\lambda_1, \dots, \lambda_r)$ for the $\lambda_i, \chi_i \in \mathbb{Z}$, and $t = (t_1, \dots, t_r)$, then

$$t^\chi = t_1^{\chi_1} t_2^{\chi_2} \dots t_r^{\chi_r} \quad \text{and} \quad z^\lambda = (z^{\lambda_1}, \dots, z^{\lambda_r}).$$

Reductive groups

Let G be a connected linear algebraic group over $\mathbb{C}$. The *radical* of G is the (unique) maximal connected solvable normal closed subgroup $R(G) \subset G$. If $R(G)$ is trivial,

G is *semisimple*. Semisimple groups turn out to be products of simple groups (with some amalgamation of finite centres), of which the classification in terms types $A_r, B_r, \dots, G_2$ is well known.

The radical $R(G)$ of a linear algebraic group G is again a linear algebraic group and as such it admits a Jordan decomposition into a semisimple and a unipotent part. The unipotent part $G^u = R^u(G)$ of the radical is called the *unipotent radical* of G. On the other hand, the semisimple part Z of the radical $R(G)$ is diagonalizable, so G acts trivially on it by conjugation. Therefore, Z is contained in the *centre* of G, $Z_G = \{z \in G : [z, g] = 1, \forall g \in G\}$. By the maximality and the connectedness assumptions on $R(G)$, it follows that $Z = Z_G^0$ is the neutral connected component of the centre Z_G.

The group G is *reductive* if its unipotent radical G^u is trivial or, equivalently if it radical $R(G)$ is a complex algebraic torus. More precisely, the radical $R(G)$ of a reductive group G is the torus $Z = Z_G^0$. Therefore, the *derived subgroup* of G

$$G' = \{[g, h] : g, h \in G\} = G/Z$$

is semisimple, and G decomposes as a semidirect product

$$G = G'Z.$$

In general, every linear algebraic group G over $\mathbb{C}$ admits a *Levi decomposition*

$$G = LG^u$$

as a semidirect product of a reductive subgroup L, called the *Levi factor* of G, and the unipotent radical G^u.

Amongst all the subgroups of a reductive group G, *maximal tori* and *Borel subgroups* are of particular interest. A *torus* $A \subset G$ is a subgroup of G isomorphic to a complex algebraic torus, and a *maximal torus* $T \subset G$ is one which is maximal among those with that property. A *Borel subgroup* $B \subset G$ is a maximal connected solvable subgroup of G. It is a fact that all Borel subgroups and all maximal tori are pairwise conjugate. The *rank* of the reductive group G is by definition the rank of a maximal torus $T \subset G$.

The subgroups $P \subset G$ that contain a Borel subgroup are called *parabolic subgroups*, and they are characterized by the property that the homogeneous space G/P is a projective variety. These projective varieties are called *flag varieties*, and the projective variety G/B, obtained as the quotient of G by a Borel subgroup, is maximal among them, and it is called the *complete flag variety*.

The *Levi subgroups* of a reductive group G are also important. These are the subgroups $L \subset G$ of the form $L = C_G(A)$, for $A \subset G$ any torus. Since the radical of $C_G(A)$ is equal to A, the Levi subgroups are reductive. In particular, if T is a maximal torus, then $C_G(T)$ is nilpotent and thus solvable, therefore it is a torus, and in turn $C_G(T) = T$. In general, the Levi subgroups $L \subset G$ can be characterized as those obtained from parabolic subgroups via Levi decomposition

$$P = LP^u.$$

Given a parabolic subgroup $P \subset G$ with Levi decomposition $P = LP^u$, we say that another parabolic subgroup $P^- \subset G$ is *opposite* of P if the intersection $P \cap P^-$ is the Levi subgroup L. The Levi decomposition of a Borel subgroup B is $B = TB^u$, for $T \subset G$ a maximal torus, and $B \cap B^- = T$. One last thing to note is that, since any character $P \to \mathbb{C}^*$ must send the unipotent radical to the unipotent part of $\mathbb{C}^*$, which is trivial, the set $X^*(P)$ of characters of P coincides with the set $X^*(L)$ of characters of its Levi factor. In particular, we have $X^*(B) = X^*(T)$.

Root systems

We recall now the basics of root systems.

Let $(\mathbb{E}, (\cdot,\cdot))$ be a finite-dimensional Euclidean vector space. Recall that a (crystallographic) *root system* in $\mathbb{E}$ is a finite subset $\Phi \subset \mathbb{E}$, whose elements are called *roots*, such that:

1. The roots span $\mathbb{E}$.
2. For any two roots α, β, the vector $s_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ is in Φ.
3. For any two roots α, β, the number $2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

We say that a root system is *reduced* if the only scalar multiples of a root α that belong to Φ are α and $-\alpha$. In any case, note that even if Φ is nonreduced, the only multiples of a root α that can belong to Φ are $\pm\alpha$, $\pm\frac{1}{2}\alpha$ or $\pm 2\alpha$, since

$$2\frac{(\alpha, \lambda\alpha)}{(\alpha,\alpha)} = 2\lambda \quad \text{and} \quad 2\frac{(\alpha,\lambda\alpha)}{(\lambda\alpha,\lambda\alpha)} = 2/\lambda$$

must be integers.

Given a root system Φ, one can fix a subset Φ^+ of *positive roots* which is closed under the sum and such that for each $\alpha \in \Phi$ either α or $-\alpha$ belong to Φ^+ (but not both). When Φ is reduced, the indecomposable elements of Φ^+ form the set of *simple roots* Δ, and every root $\alpha \in \Phi$ can be written as a linear combination of elements in Δ with integer coefficients. Fixing a set of simple roots Δ is equivalent to fixing the positive roots Φ^+. This choice is unique up to the action of the *Weyl group* of Φ, which is the finite group W generated by the reflections s_α. This group naturally acts on $\mathbb{E}$, and its fundamental domains are the *Weyl chambers*, which are the connected components of the complement of the union of the hyperplanes perpendicular to each root $\alpha \in \Phi$. Given a choice of simple roots Δ, the corresponding *dominant Weyl chamber* is the one defined as

$$C^+_\Delta = \{v \in \mathbb{E} : (v,\alpha) \geq 0, \forall \alpha \in \Delta\}.$$

One can also consider the *anti-dominant Weyl chamber*

$$C^-_\Delta = \{v \in \mathbb{E} : (v,\alpha) \leq 0, \forall \alpha \in \Delta\},$$

which is related to C^+_Δ by the *longest element* of W, which is the element w_0 of maximal length as a word in the s_α, for $\alpha \in \Delta$.

To any root system Φ in $(\mathbb{E}, (\cdot,\cdot))$, one can associate its *dual root system*

$$\Phi^\vee = \left\{\alpha^\vee \in \mathbb{E}^* : \alpha \in \Phi\right\},$$

where $\alpha^\vee$ is the *coroot* of α, defined as

$$\langle \nu, \alpha^\vee\rangle = 2\frac{(\nu,\alpha)}{(\alpha,\alpha)}$$

for any $\nu \in \mathbb{E}$. Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing of $\mathbb{E}$ and $\mathbb{E}^*$. From the definition of a root system, it is clear that this duality pairing restricts to a pairing $\langle\cdot,\cdot\rangle : \Phi\times\Phi^\vee \to \mathbb{Z}$. When Φ is reduced, the *simple coroots* $\Delta^\vee$, which are the duals of the simple roots, give the simple roots for the root system $\Phi^\vee$.

The *root and weight lattices* of a root system Φ are, respectively

$$\begin{aligned}\mathcal{R}(\Phi) &= \mathbb{Z}\langle\Phi\rangle \subset \mathbb{E}\\ \mathcal{P}(\Phi) &= \left\{\nu \in \mathbb{E} : \langle \nu, \alpha^\vee\rangle \in \mathbb{Z}, \forall \alpha \in \Phi\right\}.\end{aligned}$$

We can also consider the *coroot and coweight lattices*, respectively,

$$\begin{aligned}\mathcal{R}^\vee(\Phi) &= \mathcal{R}(\Phi^\vee) = \mathbb{Z}\langle\Phi^\vee\rangle \subset \mathbb{E}^*\\ \mathcal{P}^\vee(\Phi) &= \mathcal{P}(\Phi^\vee) = \{\nu \in \mathbb{E}^* : \langle \alpha, \nu\rangle \in \mathbb{Z}, \forall \alpha \in \Phi\}.\end{aligned}$$

Note that the pairing $\langle\cdot,\cdot\rangle$ defines a perfect pairing between the root lattice and the coweight lattice, and between the weight lattice and the coroot lattice.

The elements of the intersection $\mathcal{P}_+(\Delta) = \mathcal{P}(\Phi) \cap C^+_\Delta$ are called the *dominant weights* of Φ. This intersection is a semigroup spanned by some elements $\omega_1, \ldots, \omega_n$ called the *fundamental dominant weights*. That is, we have

$$\begin{aligned}\mathcal{P}_+(\Delta) &= \mathbb{Z}_+\langle\omega_1, \ldots, \omega_n\rangle,\\ \mathcal{P}(\Phi) &= \mathbb{Z}\langle\omega_1, \ldots, \omega_n\rangle.\end{aligned}$$

Dually, and assuming that $\Delta^\vee$ defines a set of simple coroots, we can consider the intersection

$$\mathcal{P}^\vee_+(\Delta) = \mathcal{P}^\vee(\Phi) \cap C^+_{\Delta^\vee} = \left\{\lambda \in \mathcal{P}^\vee(\Phi) : \langle\lambda,\alpha\rangle \geq 0, \forall\alpha \in \Delta\right\}.$$

This is the set of *dominant coweights* of Φ. Again, this set is a semigroup spanned by some elements $\lambda_1, \ldots, \lambda_n$ called the *fundamental dominant coweights*, and

$$\begin{aligned}\mathcal{P}^\vee_+(\Delta) &= \mathbb{Z}_+\langle\lambda_1, \ldots, \lambda_n\rangle,\\ \mathcal{P}(\Phi) &= \mathbb{Z}\langle\lambda_1, \ldots, \lambda_n\rangle.\end{aligned}$$

A choice of the simple roots Δ also determines an order on $\mathbb{E}$, and thus also on $\mathcal{P}(\Phi)$ and $\mathcal{R}(\Phi)$, given by

$$\nu \geq \nu' \text{ if and only if } \nu - \nu' \in \mathbb{Z}_+\langle\Delta\rangle.$$

Note that, if Φ is a nonreduced root system, and α is a root with $2\alpha \in \Phi$ then $(2\alpha)^\vee = \alpha^\vee/2$. Thus, when Φ is nonreduced, in order for the above definitions to work properly, we need to define the simple roots as the union of the indecomposable elements with their positive multiples that belong to Φ. Given that definition, the dual simple roots $\Delta^\vee$ will be simple roots for $\Phi^\vee$ and we will be able to define Weyl chambers and fundamental and dominant weights and coweights just like in the reduced case. Note that this change in the definition does not change the Weyl group, since the reflection associated to α is the same that the one associated to 2α.

Finally, we consider the *multiplicative invariants* of a root system. By this we mean the ring $\mathbb{C}[e^{\mathcal{P}(\Phi)}]^W$ of W-invariants of the group algebra of the weight lattice $\mathbb{C}[e^{\mathcal{P}(\Phi)}]$. We write $e^{\mathcal{P}(\Phi)}$ in order to regard the weight lattice as a multiplicative abelian group, rather than an additive one. Now, given any element $a \in \mathbb{C}[e^{\mathcal{P}(\Phi)}]$, which is of the form

$$a = \sum_{\omega \in \mathcal{P}(\Phi)} a_\omega e^\omega,$$

we define the weights of a to be those ω such that $a_\omega \neq 0$. The maximal elements among these weights are called the *highest weights* of a. The main result here is the following; we refer to [Bou02, VI.3.4 Theorem 1] for a proof.

Proposition 1.1.1. *Let Φ be a reduced root system, with $\Delta \subset \Phi$ a choice of simple roots. Let $\omega_1, \dots, \omega_n$ be the corresponding fundamental dominant weights of Φ and, for each i, let $a_i \in \mathbb{C}[e^{\mathcal{P}(\Phi)}]^W$ be a W-invariant element with unique highest weight ω_i. Then, there is an isomorphism*

$$\mathbb{C}[e^{\mathcal{P}(\Phi)}]^W \cong \mathbb{C}[a_1, \dots, a_n].$$

The root system of a reductive group

Let G be a reductive algebraic group over $\mathbb{C}$. There is a natural root system associated to the group G, or rather, to its Lie algebra $\mathfrak{g}$. Indeed, if we take a maximal torus $T \subset G$, the *roots* of $\mathfrak{g}$ are the weights of the adjoint action of T in $\mathfrak{g}$; that is, the set of roots of $\mathfrak{g}$ is the following

$$\Phi_{\mathfrak{g}} = \{\alpha \in X^*(T) : \exists x \in \mathfrak{g} \setminus \{0\}, \mathrm{Ad}_t(x) = t^\alpha x, \forall t \in T\}.$$

For each $\alpha \in \Phi_{\mathfrak{g}}$, we denote by $\mathfrak{g}_\alpha$ the subspace of $\mathfrak{g}$ formed by the $x \in \mathfrak{g}$ such that $\mathrm{Ad}_t(x) = t^\alpha x$ for every $t \in T$. This gives a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{g}}} \mathfrak{g}_\alpha.$$

It is easy to show that $\Phi_{\mathfrak{g}}$ satisfies the axioms of a reduced root system. Moreover, the Weyl group of $\Phi_{\mathfrak{g}}$ can be identified with

$$W_T = N_G(T)/T.$$

A choice of positive roots $\Phi_{\mathfrak{g}}^+ \subset \Phi_{\mathfrak{g}}$ determines a Borel subgroup $B \subset G$ and viceversa. This Borel subgroup B is the one with Lie algebra

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{g}}^+} \mathfrak{g}_\alpha,$$

so its unipotent radical is $\mathfrak{b}^u = \bigoplus_{\alpha\in\Phi_\mathfrak{g}^+} \mathfrak{g}_\alpha$. The choice of the positive roots determines a the simple roots $\Delta_\mathfrak{g} = \{\alpha_1, \ldots, \alpha_r\}$, for r the *rank* of G. More generally, if P is any parabolic subgroup containing B, then its Lie algebra is of the form

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi_\mathfrak{g}^+\cup\Phi_I} \mathfrak{g}^\alpha,$$

for Φ_I the root system spanned by some subset of the simple roots $I \subset \Delta_\mathfrak{g}$. If $P = LP^u$ is the Levi decomposition of P, the Lie algebras of these components are

$$\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi_I} \mathfrak{g}^\alpha \quad \text{and} \quad \mathfrak{p}^u = \bigoplus_{\alpha\in\Phi_\mathfrak{g}^+\setminus\Phi_I^+} \mathfrak{g}_\alpha.$$

The root lattice $\mathcal{R}(\Phi_\mathfrak{g})$ is equal to $X^*(T^{ad})$ the set of characters of the torus $T^{ad} = T/Z_G$, the maximal torus of the adjoint form of $\mathfrak{g}$, $G^{ad} = G/Z_G$. Moreover, if G is semisimple, the weight lattice $\mathcal{P}(\Phi_\mathfrak{g})$ is equal to $X^*(T^{sc})$, for T^{sc} the maximal torus of G^{sc}, the simply-connected group isogenous to G. Dually, the coweight lattice $\mathcal{P}^\vee(\Phi_\mathfrak{g})$ is equal to the cocharacter lattice $X_*(T^{ad})$, while the coroot lattice $\mathcal{R}^\vee(\Phi_\mathfrak{g})$ is equal to $X_*(T^{sc})$.

If G is semisimple, we have the natural exact sequences

$$1 \longrightarrow \pi_1(G) \longrightarrow T^{sc} \longrightarrow T \longrightarrow 1,$$

$$1 \longrightarrow Z_G \longrightarrow T \longrightarrow T^{ad} \longrightarrow 1,$$

which in turn induce the exact sequences

$$1 \longrightarrow X_*(T^{sc}) = \mathcal{R}^\vee(\Phi_\mathfrak{g}) \longrightarrow X_*(T) \longrightarrow \pi_1(G) \longrightarrow 1,$$

$$1 \longrightarrow X_*(T) \longrightarrow X_*(T^{ad}) = \mathcal{P}^\vee(\Phi_\mathfrak{g}) \longrightarrow Z_G \longrightarrow 1.$$

Therefore, we can recover the centre Z_G and the fundamental group $\pi_1(G)$ as the quotients

$$Z_G = \mathcal{P}^\vee(\Phi_\mathfrak{g})/X_*(T), \quad \text{and} \quad \pi_1(G) = X_*(T)/\mathcal{R}^\vee(\Phi_\mathfrak{g}).$$

Root groups and pinnings

Let G be a reductive group over $\mathbb{C}$ and $T \subset G$ a maximal torus. Consider $\Phi_\mathfrak{g}$ the corresponding root system. For any root $\alpha \in \Phi_\mathfrak{g}$, we can consider a non-zero element $e_\alpha \in \mathfrak{g}_\alpha$. Such an element determines an embedding $u_\alpha : \mathbb{C} \to G$ with the property that

$$tu_\alpha(z)t^{-1} = u_\alpha(t^\alpha z),$$

for every $t \in T$ and $z \in \mathbb{C}$, by putting $u_\alpha(z) = \exp(ze_\alpha)$. The image $U_\alpha = u_\alpha(\mathbb{C})$ is called the corresponding *root group*. Note that e_α can be recovered from u_α.

A *pinning* of G is a tuple $(B, T, \{e_\alpha : \alpha \in \Delta_\mathfrak{g}\})$, where $B \subset G$ is a Borel subgroup, $T \subset B$ a maximal torus, $\Delta_\mathfrak{g}$ is the set of simple roots determined by B and T, and $e_\alpha \in \mathfrak{g}_\alpha$ is a non-zero vector. Pinnings rigidify the group G, meaning that the only automorphism of G preserving a pinning is the identity. Moreover, any two pinnings of G differ by an inner automorphism; that is, if $(B, T, \{e_\alpha\})$ and $(B', T', \{e'_\alpha\})$ are different pinnings, then there exists some $g \in G$ such that

$$B' = gBg^{-1}, \;\; T' = gTg^{-1}, \;\; \text{and} \;\; \{e'_\alpha\} = \mathrm{Ad}_g\{e_\alpha\}.$$

Moreover, the inner automorphism given by conjugation by g is uniquely determined by the two pinnings; meaning that a different g' giving the same would be in the centre.

Some representation theory

A (rational) *representation* of a linear algebraic group G, or a G-*module* is a (finite dimensional) complex vector space V endowed with a homomorphism of linear algebraic groups $G \to GL(V)$. A G-module is *simple* or *irreducible* if it has no proper nontrivial G-submodule. If G is a reductive group, then any G-module V decomposes as a sum of simple G-modules. In fact, this property characterizes reductive groups [Mil17, Theorem 22.42].

Given any two G-modules V and W, the *multiplicity* of V in W is the number

$$m_V(W) = \dim \mathrm{Hom}_G(V, W).$$

If G is reductive, then any G-module W decomposes as

$$W = \bigoplus_{V \text{ simple}} m_V(W)V.$$

If Σ is a G-variety, then the ring of regular functions $\mathbb{C}[\Sigma]$ is naturally a G-module, and we abbreviate, for any G-module V,

$$m_V(\Sigma) = m_V(\mathbb{C}[\Sigma]).$$

If $H \subset G$ is a closed subgroup of G, we can consider the *induction functor* Ind_H^G : H-modules $\to$ G-modules and the *restriction functor* Res_H^G : G-modules $\to$ H-modules. For any G-module V, the restriction $\mathrm{Res}_H^G(V)$ is just the same vector space, now regarded as an H-module. On the other hand, if V is an H-module, we put $\mathrm{Ind}_H^G(V) = \mathrm{Hom}_H(G, V)$. In particular, note that $\mathrm{Ind}_H^G(\mathbb{C}) = \mathbb{C}[G]^H = \mathbb{C}[G/H]$. *Frobenius reciprocity* implies that these two functors are adjoint, meaning that, for any G-module V and any H-module W, there is an isomorphism

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G V, W).$$

As a particular case we have

$$\mathrm{Hom}_G(V, \mathbb{C}[G/H]) = \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(\mathbb{C})) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G V, \mathbb{C}) = (V^*)^H.$$

Therefore,

$$m_V(G/H) = \dim(V^*)^H.$$

The Lie–Kolchin theorem asserts that a representation of a solvable linear algebraic group over $\mathbb{C}$ has an invariant line. In particular, if G is a reductive group, then every G-module V contains at least a B-*eigenvector*, that is, an element $v \in V$ such that there exists a character $\chi : B \longrightarrow \mathbb{C}^*$ such that $b \cdot v = \chi(b)v$. We say that such a χ is a *weight* of V. If a G-module is generated by a B-eigenvector v as a G-module, then v is called a *highest weight vector* of V, and its weight χ is the *highest weight* of V. Indeed, this weight is the highest in the sense that any for any other weight η of V, we have $\chi \geq \eta$ (that is, $\chi - \eta$ is a positive integer combination of simple roots). Conversely, one of the most important results in the representation theory of reductive groups is that, if $T \subset B$ is a maximal torus, for any dominant character $\chi \in X^*(T)_+$ there exists a unique, up to isomorphism, G-module V_χ with highest weight χ (see [Hum75, Section 31]). A nice consequence of this is that the ring of regular functions of a reductive group splits as a $(G \times G)$-module as

$$\mathbb{C}[G] = \bigoplus_{V \text{ simple}} \operatorname{End} V = \bigoplus_{V \text{ simple}} V^* \otimes V = \bigoplus_{\chi \in X^*(T)} V_\chi^* \otimes V_\chi.$$

Root data and the Langlands dual group

A (reduced) *root datum* is a tuple $(X^*, \Phi, X_*, \Phi^\vee)$, with X^* and X_* lattices in duality with respect to a perfect pairing $\langle \cdot, \cdot \rangle : X^* \times X_* \longrightarrow \mathbb{Z}$ and $\Phi \subset X^*$ and $\Phi^\vee \subset X_*$ subsets with a fixed bijection $\Phi \longrightarrow \Phi^\vee$, which we denote by $\alpha \mapsto \alpha^\vee$, such that Φ is a reduced root system on $X^* \otimes_{\mathbb{Z}} \mathbb{R}$ and $\Phi^\vee$ is the dual root system. If we take a set of simple roots $\Delta \subset \Phi$ and its dual $\Delta^\vee \subset \Phi^\vee$, the tuple $(X^*, \Delta, X_*, \Delta^\vee)$ is called a *based root datum*.

For any reductive group G and any choice of a Borel subgroup $B \subset G$ and of a maximal torus $T \subset B \subset G$, we can take $\Phi_T \subset X^*(T)$ the root system of T and $\Delta_{B,T} \subset \Phi_T$ the simple roots determined by B. The tuple $\Psi(G, B, T) = (X^*(T), \Delta_{B,T}, X_*(T), \Delta_{B,T}^\vee)$ is called the *based root datum of* (G, B, T). One can also give a natural notion of a morphism of based root data. For any other choice of Borel subgroup B' and maximal torus T' the root data $\Psi(G, B, T)$ and $\Psi(G, B', T')$ are canonically isomorphic, and one can define the *based root datum of* G to be the projective limit $\Psi(G)$ of all the based root data for different choices of B. There is a natural isomorphism $\operatorname{Aut}(\Psi) \cong \operatorname{Out}(G)$.

Based root data are well known to classify reductive groups and thus, for any reductive group G with based root datum $\Psi(G) = (X^*, \Delta, X_*, \Delta^\vee)$, we can define the (Langlands) *dual group* of G to be the reductive group $\check{G}$ with based root data $\Psi(\check{G}) = \Psi(G)^\vee = (X_*, \Delta^\vee, X^*, \Delta)$. The *geometric Satake correspondence* [Gin00, MV07] recovers $\check{G}$ as the Tannaka group of the neutral Tannakian category of perverse sheaves on the affine Grassmannian Gr_G (defined below in Section 1.7).

Tits systems

The role played by the root system, the Borel subgroup and the maximal torus of a reductive group is so important in the determination of its structure that these notions can be abstracted and applied in a wider spectrum of cases. This is the theory of *Tits systems* or *BN-pairs*.

A *Tits system* $(\mathfrak{G}, \mathfrak{B}, \mathfrak{N}, \mathfrak{S})$ consists of a group $\mathfrak{G}$, subgroups $\mathfrak{B}$ and $\mathfrak{N}$ and a finite subset $\mathfrak{S} \subset \mathfrak{N}/(\mathfrak{B} \cap \mathfrak{N})$ satisfying the following axioms:

1. $\mathfrak{B} \cap \mathfrak{N}$ is a normal subgroup of $\mathfrak{N}$ and $\mathfrak{S}$ generates the *Weyl group* $W = \mathfrak{N}/\mathfrak{B} \cap \mathfrak{N}$.

2. $\mathfrak{B}$ and $\mathfrak{N}$ generate $\mathfrak{G}$ as a group,

3. for any $s \in \mathfrak{S}$, $s\mathfrak{B}s^{-1} \not\subset \mathfrak{B}$, and

4. for every $w \in W$ and $s \in \mathfrak{S}$, if $n_w, n_s \in \mathfrak{N}$ are representatives of w and s, respectively, we have
$$(\mathfrak{B} n_s \mathfrak{B})(\mathfrak{B} n_w \mathfrak{B}) \subset (\mathfrak{B} n_w \mathfrak{B}) \cup (\mathfrak{B} n_s n_w \mathfrak{B}).$$

The paradigmatical example is of course taking $\mathfrak{G} = G$ to be a reductive group, $\mathfrak{B} = B \subset G$ a Borel subgroup, $\mathfrak{N} = N_G(T)$ the normalizer of a maximal torus $T \subset G$, so that $W = \mathfrak{N}/(\mathfrak{B} \cap \mathfrak{N}) = \mathfrak{N}/T$ is the Weyl group, and $\mathfrak{S}$ is the set of the reflections s_α, for $\alpha \in \Delta_\mathfrak{g}$ the simple roots.

The axioms of a Tits system allow to generalize a lot of properties of reductive groups to any Tits system. The *formal loop groups* considered in this document are nice examples of these groups that, although they are not reductive, admit the structure of a Tits system. Of all the properties that one can generalize, we are mainly interested in the Bruhat decomposition: if $(\mathfrak{G}, \mathfrak{B}, \mathfrak{N}, \mathfrak{S})$ is a Tits system, then
$$\mathfrak{G} = \bigsqcup_{w \in W} \mathfrak{B} w \mathfrak{B}.$$

1.2 SOME GENERALITIES ON INVOLUTIONS

Involutions and the twisted conjugation action

Let G be a reductive algebraic group over $\mathbb{C}$. By an *involution* of G we mean an order 2 automorphism $\theta \in \mathrm{Aut}_2(G)$. To any such involution we can associate the subgroups
$$G^\theta = \{g \in G : \theta(g) = g\},$$
$$G_\theta = \left\{g \in G : g\theta(g)^{-1} \in Z_G\right\}.$$

The group G^θ is the group of fixed points of θ, and we can also consider the group $G_0^\theta = (G^\theta)^0$, which is the neutral connected component of G^θ. If G is of simply-connected type, then G^θ is connected, so $G_0^\theta = G^\theta$ (this is not immediate, and we refer to [Ste68, Theorem 8.1] for a proof). The group G_θ is in fact equal to the normalizer $N_G(G^\theta)$ of G^θ. The inclusion $G_\theta \subset N_G(G^\theta)$ is immediate, while the other inclusion follows from the fact that elements of the form $g\theta(g)^{-1}$ act trivially by conjugation on G^θ and, if we take the *Cartan decomposition*
$$\mathfrak{g} = \mathfrak{g}^\theta \oplus \mathfrak{m}^\theta$$

into $\mathfrak{g}^\theta = \mathrm{Lie}(G^\theta)$ the +1-eigenspace of θ and $\mathfrak{m}^\theta$ the −1-eigenspace then, for every $x \in \mathfrak{m}^\theta$ and every $g \in N_G(G^\theta)$, we have

$$-\mathrm{Ad}_g(x) = \theta(\mathrm{Ad}_g(x)) = -\mathrm{Ad}_{\theta(g)}(x),$$

and thus $\mathrm{Ad}_{g\theta(g)^{-1}}(x) = x$, so $g\theta(g)^{-1} \in Z_G$.

An involution θ of G determines the θ-*twisted conjugation action* of G on itself

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, s) &\longmapsto g *_\theta s = gs\theta(g)^{-1}. \end{aligned}$$

For each $s \in G$ we denote by $\tau^\theta_s : G \to G$ the map $\tau^\theta_s(g) = g *_\theta s$ sending G to the θ-twisted G-orbit $\tau^\theta_s(G) = G *_\theta s$. We also denote these θ-twisted orbits by $M^\theta_s = \tau^\theta_s(G)$. In particular, we denote $\tau^\theta = \tau^\theta_1$ and $M^\theta = \tau^\theta(G)$. One can easily check that the isotropy subgroup of $s \in G$ is the fixed point subgroup G^{θ_s} of the automorphism

$$\theta_s = \mathrm{Int}_s \circ \theta.$$

Here, Int_s stands for the automorphism of G given by conjugation by s; the elements of this form are the *inner automorphisms*, that form a subgroup $\mathrm{Int}(G) \subset \mathrm{Aut}(G)$.

From the above, we conclude that the θ-twisted orbits M^θ_s are homogeneous spaces of the form G/G^{θ_s}. Moreover, this identification is G-equivariant for the θ-twisted conjugation action on M^θ_s and the natural left multiplication action on G/G^{θ_s}. In particular, M^θ is naturally G-isomorphic to G/G^θ. We can pose the question of for which $s \in G$ the automorphism θ_s is an involution. A simple computation shows that $\theta_s^2 = \mathrm{id}_G$ if and only if $s\theta(s)$ is an element of the centre Z_G. The set of such s is denoted by

$$S_\theta = \{s \in G : s\theta(s) \in Z_G\}.$$

For $s \in S_\theta$, the θ-twisted orbit M^θ_s gets explicitly identified with M^{θ_s} as

$$M^\theta_s = M^{\theta_s} s.$$

In particular, note that if $s \in M^\theta$, then $M^{\theta_s}s = M^\theta_s = M^\theta$, so for $s \in M^\theta$ the homogeneous spaces G/G^θ and G/G^{θ_s} can be identified. Moreover, if $s = g\theta(g)^{-1}$, we can write $\theta_s = \mathrm{Int}_g \circ \theta \circ \mathrm{Int}_g^{-1}$ and $G^{\theta_s} = \mathrm{Int}_g(G^\theta)$.

It makes sense then to define an equivalence relation $\sim$ on $\mathrm{Aut}_2(G)$:

$$\theta \sim \theta' \text{ if and only if there exists } \alpha \in \mathrm{Int}(G) \text{ such that } \theta' = \alpha \circ \theta \circ \alpha^{-1},$$

and describe the quotient set $\mathrm{Aut}_2(G)/\sim$. We recall the description given in [GPR19] in terms of what they call the "clique map".

The clique map

The natural projection $\pi : \mathrm{Aut}_2(G) \to \mathrm{Out}_2(G) := \mathrm{Aut}_2(G)/\mathrm{Int}_2(G)$ factors through $\mathrm{Aut}_2(G)/\sim$. Indeed, this follows from the fact that

$$\mathrm{Int}_g \circ \theta \circ \mathrm{Int}_g^{-1} = \mathrm{Int}_{g\theta(g)^{-1}} \circ \theta.$$

Therefore, we obtain a surjective map

$$\mathrm{cl} : \mathrm{Aut}_2(G)/\!\sim\; \longrightarrow \mathrm{Out}_2(G),$$

called the *clique map*. The inverse image $\mathrm{cl}^{-1}(a)$ of a class $a \in \mathrm{Out}_2(G)$ is called the *clique* of a. It is easy to check that for any $\theta \in \pi^{-1}(a)$ the clique $\mathrm{cl}^{-1}(a)$ is in bijection with the orbit set $S_\theta/(G \times Z_G)$, where $(G \times Z_G)$ acts on S_θ through the natural extension of the θ-twisted conjugation action

$$\begin{aligned}(G \times Z_G) \times G &\longrightarrow G\\ ((g, z), s) &\longmapsto (g, z) *_\theta s = zgs\theta(g)^{-1}.\end{aligned}$$

Nonabelian group cohomology

The θ-twisted conjugation action clearly preserves S_θ and also, for each $z \in Z_G$, the subset

$$S_z^\theta = \{s \in G : s\theta(s) = z\} \subset S_\theta.$$

In particular, it preserves the subset of “anti-fixed" points of θ,

$$S^\theta = S_1^\theta = \left\{s \in G : s = \theta(s)^{-1}\right\}.$$

Now, as explained in [GPR19], the orbit sets S^θ/G and $S_\theta/(G \times Z_G)$ can be interpreted in terms of *nonabelian group cohomology* [Ser02, Section 5.1]. We recall the main definition here.

Definition 1.2.1. Let Γ and A be abstract groups with an action of Γ on A. We denote $H^0(\Gamma, A) = A^\Gamma$ the set of fixed points. A 1-*cocycle* of Γ in A is a map $\gamma \mapsto a_\gamma$ from Γ to A such that

$$a_{\gamma\gamma'} = a_\gamma(\gamma \cdot a_{\gamma'})$$

for any $\gamma, \gamma' \in \Gamma$. The set of 1-cocycles is denoted by $Z^1(\Gamma, A)$. We obtain $H^1(\Gamma, A)$ the *first cohomology set* of Γ in A as the quotient of $Z^1(\Gamma, A)$ by the equivalence relation

$$a \sim a' \text{ if and only if there exists some } b \in A \text{ such that } a'_\gamma = ba_\gamma(\gamma \cdot b)^{-1}.$$

When A is an abelian group, we can extend this definition to consider higher order cocycles and cohomology groups, and recover the well-known theory of group cohomology. For convenience, we recover the definition of order 2 group cohomology from this point of view.

Definition 1.2.2. Suppose that A is abelian. A 2-*cocycle* of Γ in A is a map $(\gamma_1, \gamma_2) \mapsto c_{\gamma_1,\gamma_2}$ form $\Gamma \times \Gamma$ to A such that $c_{\gamma,1} = c_{1,\gamma} = 1$ and

$$(\gamma \cdot c_{\gamma',\gamma''})c_{\gamma,\gamma'\gamma''} = c_{\gamma,\gamma'}c_{\gamma\gamma',\gamma''}$$

for any $\gamma, \gamma', \gamma'' \in \Gamma$. We denote the group of 2-cocycles by $Z^2(\Gamma, A)$. Now, for any map $a : \Gamma \to A$ we can consider the corresponding 2-*coboundary* $da \in Z^2(\Gamma, A)$ such that

$$(da)_{\gamma_1,\gamma_2} = (\gamma_1 \cdot a(\gamma_2))a(\gamma_1\gamma_2)^{-1}a(\gamma_1)$$

for any $\gamma_1, \gamma_2 \in \Gamma$. The group of 2-coboundaries is denoted by $B^2(\Gamma, A)$ and we recover the *second cohomology group* of Γ in A as the quotient

$$H^2(\Gamma, A) = Z^2(\Gamma, A)/B^2(\Gamma, A).$$

Taking $\Gamma = \mathbb{Z}/2 = \{\pm 1\}$, $A = G$ and the action of $\mathbb{Z}/2$ on G the one defined by $-1 \mapsto \theta$, we can consider the corresponding cohomology set, which we denote by $H^1_\theta(\mathbb{Z}/2, G)$. We can also take $A = G^{ad} = G/Z_G$, and the cohomology set $H^1_\theta(\mathbb{Z}/2, G^{ad})$. Finally, if we take $A = Z_G$, we can also define the second cohomology group $H^2_\theta(\mathbb{Z}/2, Z_G)$. As is usual in cohomology theories, one can show that the short exact sequence $1 \to Z_G \to G \to G^{ad} \to 1$ induces a long exact sequence in cohomology

$$H^1_\theta(\mathbb{Z}/2, Z_G) \longrightarrow H^1_\theta(\mathbb{Z}/2, G) \longrightarrow H^1_\theta(\mathbb{Z}/2, G^{ad}) \longrightarrow H^2_\theta(\mathbb{Z}/2, Z_G).$$

We can make sense of this by noting that the nonabelian group cohomology sets are pointed sets, where the special element is the constant cocycle with value 1.

Observe that any map from $\mathbb{Z}/2$ is determined by the image of -1, so one can easily show that the cocycles for the cohomology sets considered above are

$$\begin{aligned} Z^1_\theta(\mathbb{Z}/2, Z_G) &= \{z \in Z_G : z\theta(z) = 1\} = Z_G \cap S^\theta \\ Z^1_\theta(\mathbb{Z}/2, G) &= \{s \in G : s\theta(s) = 1\} = S^\theta \\ Z^1_\theta(\mathbb{Z}/2, G^{ad}) &= \{s \in G : s\theta(s) \in Z_G\} = S_\theta \\ Z^2_\theta(\mathbb{Z}/2, Z_G) &= \{z \in Z_G : \theta(z) = z\} = Z_G \cap G^\theta. \end{aligned}$$

Now, the equivalence relation in the sets of 1-cocycles can inmediately be understood as the relation of being in the same θ-twisted G-orbit and the coboundary group $H^2_\theta(\mathbb{Z}/2, Z_G)$ is identified with the subgroup of elements of the form $z\theta(z)$ for $z \in Z_G$. Thus, we get

$$\begin{aligned} H^1_\theta(\mathbb{Z}/2, Z_G) &= \{z \in Z_G : z\theta(z) = 1\} = (Z_G \cap S^\theta)/G \\ H^1_\theta(\mathbb{Z}/2, G) &= \{s \in G : s\theta(s) = 1\} = S^\theta/G \\ H^1_\theta(\mathbb{Z}/2, G^{ad}) &= \{s \in G : s\theta(s) \in Z_G\} = S_\theta/(G \times Z_G) = \mathrm{cl}^{-1}(\pi(\theta)) \\ H^2_\theta(\mathbb{Z}/2, Z_G) &= \{z \in Z_G : \theta(z) = z\} = (Z_G \cap G^\theta)/\{z\theta(z) : z \in Z_G\}. \end{aligned}$$

In these terms, we can understand the maps giving the above long exact sequence as follows. The map $H^1_\theta(\mathbb{Z}/2, Z_G) \to H^1_\theta(\mathbb{Z}/2, G)$ sends any θ-twisted G-orbit $G *_\theta z$ of an element $z \in Z_G \cap S^\theta$ to itself. The map $H^1_\theta(\mathbb{Z}/2, G) \to H^1_\theta(\mathbb{Z}/2, G^{ad})$ sends a θ-twisted G-orbit $G *_\theta s$ to the bigger $G \times Z_G$-orbit $(G \times Z_G) *_\theta s$. Finally, the map $H^1_\theta(\mathbb{Z}/2, G) \to H^2_\theta(\mathbb{Z}/2, Z_G)$ sends an orbit $(G \times Z_G) *_\theta s$ to the class of the element $s\theta(s) \in Z_G$. Indeed, this is well defined since, if $s' = zgs\theta(g)^{-1}$ for some $z \in Z_G$ and $g \in G$, then $s'\theta(s') = z\theta(z)s\theta(s)$.

We end our comments on nonabelian group cohomology by mentioning that it follows from a theorem of Richardson [Ric82a] that there is a finite number of θ-twisted G-orbits on S^θ, and thus all the cohomology sets mentioned above are finite.

Real forms

Let $G_{\mathbb{R}}$ be a linear algebraic group over $\mathbb{R}$. We define its *complexification* to be the base change $G_{\mathbb{C}} = G_{\mathbb{R}} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$, which is a linear algebraic group over $\mathbb{C}$.

Definition 1.2.3. Let G be a linear algebraic group over $\mathbb{C}$. A *real form* of G is a linear algebraic group $G_{\mathbb{R}}$ over $\mathbb{R}$ endowed with an isomorphism $G_{\mathbb{C}} \xrightarrow{\sim} G$.

Equivalently, a real form of a complex group G is given by a *conjugation* $\sigma \in \operatorname{Conj}(G)$. This is an involutive anti-holomorphic automorphism of G. The real form $G_{\mathbb{R}}$ is recovered as the fixed point subgroup $G_{\mathbb{R}} = G^{\sigma}$. By similar arguments to the ones given above, two real forms G^{σ} and $G^{\sigma'}$ are conjugated by some element $g \in G$ if and only if the two conjugations σ and σ' are conjugated by the inner automorphism Int_g. Therefore, we say that two real forms σ and σ' are equivalent if and only if there exists some inner automorphism $\alpha \in \operatorname{Int}(G)$ such that

$$\sigma' = \alpha \circ \sigma \circ \alpha^{-1}.$$

Up to this notion of equivalence, real forms are classified by the nonabelian group cohomology set $H^1_{\sigma}(\mathbb{Z}/2, G^{ad})$. Noting that the Galois group of $\mathbb{C}$ over $\mathbb{R}$ is $\operatorname{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$, this cohomology set is by construction the *nonabelian Galois cohomology set*

$$H^1(\mathbb{C}/\mathbb{R}, G^{ad}) := H^1_{\sigma}(\operatorname{Gal}(\mathbb{C}/\mathbb{R}), G^{ad}).$$

See [Ser02] for further reference on Galois cohomology.

Definition 1.2.4. A *compact real form* $G_{\mathbb{R}}$ of a complex linear algebraic group G is a real form $G_{\mathbb{R}}$ that is compact and meets every connected component of G. Equivalently, if $G_{\mathbb{R}}$ is a compact real form of G and $G_{\mathbb{R}} = G^{\sigma}$ for some $\sigma \in \operatorname{Conj}(G)$, we say that σ is a compact real form of G.

Every complex reductive group G admits a *maximal compact subgroup* $K \subset G$, which is in fact a real form of G, called the *compact real form*. We denote by σ_K the corresponding conjugation. Every other compact real form is equivalent to this one.

Given any involution $\theta \in \operatorname{Aut}_2(G)$, we can choose a maximal compact subgroup $K \subset G$ that is θ-stable, that is, such that $\theta(K) \subset K$. Equivalently, we can choose σ_K commuting with θ, that is $\sigma_K \circ \theta = \theta \circ \sigma_K$. The composition $\sigma_\theta = \sigma_K \circ \theta$ is another real form, which we call the *Cartan real form* associated to θ. Reciprocally, given any real form σ we can find some involution $\theta_\sigma \in \operatorname{Aut}_2(G)$, called the *Cartan involution* associated to σ, commuting with σ, and such that $\sigma \circ \theta_\sigma$ is a compact real form. Moreover, this θ_σ is unique up to conjugation by inner automorphisms. This gives the *Cartan classification of real forms*, which cohomologically can be written as a bijection

$$H^1_{\sigma}(\mathbb{Z}/2, G^{ad}) \cong H^1_{\theta}(\mathbb{Z}/2, G^{ad}).$$

More generally, in [AT18], Adams and Taïbi show the existence of a bijection

$$H^1_{\sigma}(\mathbb{Z}/2, G) \cong H^1_{\theta}(\mathbb{Z}/2, G).$$

We refer to Section 10 of their paper for tables with $H^1_{\theta}(\mathbb{Z}/2, G)$ computed for G semisimple simply-connected.

Examples

Example 1.2.5 (The diagonal case)**.** Groups with involution are in a certain sense a generalization of groups. Indeed, if G is any linear algebraic group, we can consider the pair $(G \times G, \Theta)$, where $\Theta \in \mathrm{Aut}_2(G \times G)$ is the involution $\Theta(g_1, g_2) = (g_2, g_1)$. Let us denote by $\Delta : G \to G \times G$ the *diagonal* map $\Delta(g) = (g, g)$ and by $\tilde{\Delta}$ the *antidiagonal* $\tilde{\Delta}(g) = (g, g^{-1})$. The fixed point subgroup $(G \times G)^{\Theta}$ is the diagonal $\Delta(G)$, which can be naturally identified with G, whereas the orbit M^{Θ} is the antidiagonal $\tilde{\Delta}(G)$, which is again identified with G. The set of anti-fixed points S^{Θ} is easily shown to be equal to M^{Θ}. We also have

$$(G \times G)_{\Theta} = \{(zg, g) : g \in G\}, \text{ and } S_{\Theta} = \left\{(zg, g^{-1}) : g \in G\right\}.$$

Therefore, the orbit sets $S^{\Theta}/(G \times G)$ and $S_{\Theta}/(Z \times G \times G)$ are just singletons. The real form of $G \times G$ corresponding to Θ is simply the group G regarded as a real group.

Example 1.2.6 ($G = \mathrm{SL}_n(\mathbb{C}), n > 2$)**.** In this case we have $\mathrm{Out}(G) \cong \mathbb{Z}/2$, so we distinguish two cases for $a \in \mathrm{Out}_2(G)$, $a = 1$ and $a = -1$. For $a = 1$ we have

$$\mathrm{cl}^{-1}(1) = \left\{\theta_{p,q} : 0 \le p \le q \le n, p + q = n\right\},$$

with $\theta_{p,q}(g) = I_{p,q} g I_{p,q}$, for

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix},$$

where I_p denotes the identity matrix of rank p. The compact real form of $\mathrm{SL}_n(\mathbb{C})$ is $\mathrm{SU}(n)$, which sits inside $\mathrm{SL}_n(\mathbb{C})$ as the fixed points of the conjugation $\sigma_K(g) = (g^{\dagger})^{-1}$, where † stands for taking transpose and complex-conjugation. Therefore, the real forms corresponding to the involution $\theta_{p,q}$ is $\sigma_{p,q}(g) = I_{p,q}(g^{\dagger})^{-1} I_{p,q}$. The corresponding groups of fixed points are

$$G^{\theta_{p,q}} = S(\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})) \quad \text{and} \quad G^{\sigma_{p,q}} = \mathrm{SU}(p, q).$$

Here, the S stands for taking the subset of matrices of determinant equal to 1. Consider now the case $a = -1$. The clique $\mathrm{cl}^{-1}(-1)$ consists of a sigle element if n is odd and of two elements if $n = 2m$ is even. In both cases we have the involution $\theta_0(g) = (g^T)^{-1}$, which corresponds to the split real form $\sigma_0(g) = \bar{g}$. The fixed points subgroups are

$$G^{\theta_0} = \mathrm{SO}_n(\mathbb{C}) \quad \text{and} \quad G^{\sigma_0} = \mathrm{SL}_n(\mathbb{R}).$$

When $n = 2m$ is even, we also have the involution $\theta_1(g) = J_m \theta_0(g) J_m^{-1}$, where J_m is the symplectic matrix

$$J_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

so the real form associated to it is $\sigma_1(g) = J_m \bar{g} J_m^{-1}$. The fixed points subgroups are

$$G^{\theta_1} = \mathrm{Sp}_{2m}(\mathbb{C}) \quad \text{and} \quad G^{\sigma_1} = \mathrm{SU}^*(2m).$$

We refer to Table 26.3 in [Tim11] for a complete classification of the involutions of the simple groups.

1.3 SPLIT TORI AND RESTRICTED ROOTS

Split tori

Let G be a reductive algebraic group over $\mathbb{C}$ and $\theta \in \mathrm{Aut}_2(G)$ an involution. We say that a torus $A \subset G$ is θ-*split* if $\theta(a) = a^{-1}$ for every $a \in A$, and we say that it is *maximal* θ-*split* if it is maximal among θ-split tori.

Remark 1.3.1. The name "θ-split" comes from the correspondence with real forms. More precisely, a *real algebraic torus* is by definition a real form of a complex algebraic torus, that is, a linear algebraic group $T_{\mathbb{R}}$ over $\mathbb{R}$ such that its complexification $T_{\mathbb{C}}$ is isomorphic to $(\mathbb{C}^*)^r$ for some $r \in \mathbb{Z}_+$. We say that the real algebraic torus $T_{\mathbb{R}}$ is $\mathbb{R}$-*split* if $T_{\mathbb{R}}$ is $\mathbb{R}$-isomorphic to $(\mathbb{R}^\times)^r$. An example of a real algebraic torus which is not $\mathbb{R}$-split is $U(1)^r$. Now, if we take σ_θ the Cartan real form associated to θ, then a θ-split torus A is just the complexification of an $\mathbb{R}$-split torus $A_{\mathbb{R}} \subset G^{\sigma_\theta}$.

A linear algebraic group $G_{\mathbb{R}}$ over $\mathbb{R}$ is $\mathbb{R}$-*split* if there exists a maximal $\mathbb{R}$-split torus $T_{\mathbb{R}} \subset G_{\mathbb{R}}$ such that its complexification $T_{\mathbb{C}}$ is a maximal torus of the complexification $G_{\mathbb{C}}$. A real form of a complex group G is *split* if the group $G_{\mathbb{R}}$ is split. Equivalently, the real form is split if there exists a θ-split torus which is a maximal torus of G, for θ the Cartan involution associated to the real form.

More generally, we say that the real group $G_{\mathbb{R}}$ is $\mathbb{R}$-*quasisplit* if it has an $\mathbb{R}$-split Borel subgroup $B_{\mathbb{R}} \subset G_{\mathbb{R}}$.

We now state some of the results of Vust [Vus74] on θ-split tori.

Proposition 1.3.2 (Vust)**.**

1. *Non-trivial* θ*-split tori exist.*

2. *Any maximal torus* T *of* G *containing a maximal* θ*-split torus is* θ*-stable, meaning that* $\theta(T) \subset T$*.*

3. *All maximal* θ*-split tori are pairwise conjugate by elements of* G^θ*.*

4. *For any maximal* θ*-split torus* $A \subset G$*, the group* G^θ *decomposes uniquely as* $G^\theta = F^\theta G^\theta_0$*, for*
$$F^\theta = A \cap G^\theta = \left\{a \in A : a^2 = 1\right\}.$$

5. *For any maximal* θ*-split torus* $A \subset G$*, the centralizer* $Z_G(A)$ *decomposes uniquely as*
$$Z_G(A) = (Z_G(A) \cap G^\theta)^0 A.$$

Related to point (4) in the proposition above, we also have the following result of Richardson [Ric82b, Lemma 8.1.(a)].

Lemma 1.3.3. *For any maximal* θ*-split torus* $A \subset G$*, the group* G_θ *decomposes uniquely as* $G_\theta = F_\theta G^\theta_0$*, for*
$$F_\theta = A \cap G_\theta = \left\{a \in A : a^2 \in Z_G\right\}.$$

Split parabolics

A parabolic subgroup $P \subset G$ is θ-*split* if P and $\theta(P)$ are opposite, that is, if $P \cap \theta(P)$ is a Levi subgroup of both P and $\theta(P)$. If $P \subset G$ is a minimal θ-split parabolic subgroup one can show [Vus74] that there exists a maximal θ-split torus A and a dominant cocharacter $\lambda \in X_*(A)$ such that P is of the form

$$P = P(\lambda) = \left\{ p \in G : \lim_{t\to 0} t^\lambda p t^{-\lambda} \in G \right\}.$$

Equivalently, if one takes $T \supset A$ a maximal torus, and considers $\Phi_\mathfrak{g}$ the root system and $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi_\mathfrak{g}} \mathfrak{g}^\alpha$ the corresponding root space decomposition, then $P(\lambda)$ is the parabolic subgroup with Lie algebra

$$\mathfrak{p}(\lambda) = \mathfrak{t} \oplus \bigoplus_{\langle\alpha,\lambda\rangle\geq 0} \mathfrak{g}^\alpha.$$

This $P(\lambda)$ admits the Levi decomposition

$$\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\langle\alpha,\lambda\rangle=0} \mathfrak{g}^\alpha = \mathrm{Lie}(Z_G(\lambda)) \quad \text{and} \quad \mathfrak{p}(\lambda)^u = \bigoplus_{\langle\alpha,\lambda\rangle>0} \mathfrak{g}^\alpha,$$

for $Z_G(\lambda) = Z_G(\{z^\lambda : z \in \mathbb{C}^*\})$ the centralizer of the uniparametric subgroup induced by λ. As a consequence, the Levi subgroup $P \cap \theta(P)$ is equal to

$$P \cap \theta(P) = Z_G(\lambda) = Z_G(A).$$

Remark 1.3.4. Following on the correspondence with real groups, from the above it follows that if we consider σ_θ the Cartan real form associated to θ, then $P \subset G$ is a θ-split parabolic subgroup if and only if it is the complexification of a parabolic subgroup $P_\mathbb{R} \subset G^{\sigma_\theta}$ containing as a maximal torus the $\mathbb{R}$-split torus $A_\mathbb{R}$ of which A is its complexification. In other words, P is θ-split if and only if the real form P^{σ_θ} is $\mathbb{R}$-split. This motivates the following.

Definition 1.3.5. An involution $\theta \in \mathrm{Aut}_2(G)$ is *quasisplit* if there exists a θ-split Borel subgroup $B \subset G$. Equivalently, θ is quasisplit if there exists a maximal θ-split torus A such that $T = Z_G(A)$ is a maximal torus of G.

Remark 1.3.6. Of course, θ is quasisplit if and only if the real group G^{σ_θ} is $\mathbb{R}$-quasisplit.

Proposition 1.3.7. *Any class* $a \in \mathrm{Out}_2(G)$ *can be represented by a quasisplit involution.*

Proof. Let $\theta_0 \in \mathrm{Aut}_2(G)$ be any involution representing a. Let $B \subset G$ be a Borel subgroup and B^- an opposite Borel subgroup of B. That is, B^- is such that $T = B \cap B^-$ is a maximal torus. Let $\Delta_\mathfrak{g}$ be the simple roots corresponding to B and T and consider a pinning of G of the form $(B^-, T, \{e_{-\alpha} : \alpha \in \Delta_\mathfrak{g}\})$. We can also consider the different pinning $(\theta_0(B), \theta_0(T), \{e_{\alpha^{\theta_0}} : \alpha \in \Delta_\mathfrak{g}\})$. These two pinnings are related by an inner automorphism; that is, there exists some $s \in G$ such that

$$B^- = s\theta_0(B)s^{-1}, \;\; T = s\theta_0(T)s^{-1}, \;\; \text{and} \;\; \{e_{-\alpha}\} = \mathrm{Ad}_g\{e_{\alpha^{\theta_0}}\}.$$

Therefore, if we put $\theta = \mathrm{Int}_s \circ \theta_0$, we have

$$B^- = \theta(B), \;\; T = \theta(T), \;\; \text{and} \;\; \{e_{-\alpha}\} = \{e_{\alpha^\theta}\}.$$

Moreover, θ is an involution since θ^2 is an inner automorphism of G with $B^- = \theta^2(B^-)$, $T = \theta^2(T)$ and $\{e_{-\alpha}\} = \left\{e_{-\alpha^{\theta^2}}\right\}$, so it must be trivial. □

A consequence of the existence of minimal θ-split parabolics is the *Iwasawa decomposition*.

Proposition 1.3.8 (Iwasawa decomposition). *Let $P \subset G$ be a minimal θ-split parabolic subgroup and P^u its unipotent radical. Then the product $G_0^\theta A P^u$ is an open subset of G.*

Proof. We begin by showing that $G_0^\theta P$ is open in G. It suffices to show that $\mathfrak{g}^\theta + \mathfrak{p} = \mathfrak{g}$. Now, since P and $\theta(P)$ are opposite, for any $a \in \mathfrak{g}$ we can write

$$a = b + \theta(c),$$

for $b, c \in \mathfrak{p}$. But then we can also write

$$a = (c + \theta(c)) + (b - c) \in \mathfrak{g}^\theta + \mathfrak{p}.$$

Finally, the Levi decomposition of P yields $P = Z_G(A)P^u$, from where it follows that

$$G_0^\theta P = G_0^\theta Z_G(A) P^u = G_0^\theta A P^u,$$

since $Z_G(A) = (G^\theta \cap Z_G(A))^0 A$. □

The restricted root system

Let $A \subset G$ be a maximal θ-split torus and $T \subset G$ a maximal θ-stable torus containing it. Let r denote the rank of T and l the rank of A. The number l is called the *rank of the symmetric variety* G/G^θ. Let us consider $\Phi_\mathfrak{g}$ the root system of $\mathfrak{g}$.

The involution θ naturally induces an involution $\chi \mapsto \chi^\theta$ on the characters $X^*(T)$. For each $\chi \in X^*(T)$ we define $\bar{\chi} = (\chi - \chi^\theta)/2$, which is a well defined element of $\mathcal{E}_T$. Now, the lattice formed by the elements $\bar{\chi}$ of this form is naturally identified with the group of characters $X^*(A)$.

It is easy to check that the involution on characters of T sends roots to roots. The elements of the set Φ_T^θ of roots fixed under this involution are called the *imaginary roots*. A choice of positive roots $\Phi_\mathfrak{g}^+ \subset \Phi_\mathfrak{g}$ can be made in such a way that if $\alpha \in \Phi_\mathfrak{g}^+$ is not an imaginary root, then α^θ is a negative root. Making this choice amounts to choosing a Borel subgroup $B \subset G$ contained in a minimal θ-split parabolic subgroup. Indeed, this is the Borel subgroup B with Lie algebra

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_\mathfrak{g}^+} \mathfrak{g}^\alpha.$$

Proposition 1.3.9. *If θ is quasisplit, then $\Phi_T^\theta = \varnothing$.*

Proof. If θ is quasisplit, we can choose B to be θ-split and, since B is minimal among parabolics, it must be of the form $P(\lambda)$ for some $\lambda \in X_*(A)$. Now, the Levi factor of B is the maximal torus T, so

$$\mathfrak{t} = \mathfrak{t} \oplus \bigoplus_{\langle\alpha,\lambda\rangle=0} \mathfrak{g}^{\alpha}$$

and we conclude that

$$\left\{\alpha \in \Phi_{\mathfrak{g}} : \langle\alpha,\lambda\rangle = 0\right\} = \varnothing.$$

Finally, if α is an imaginary root, since $\langle\alpha^{\theta},\lambda\rangle = -\langle\alpha,\lambda\rangle$, we have $\langle\alpha,\lambda\rangle = 0$. □

Consider now the vector space $\mathcal{E}_\theta = X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}$, which is naturally a subspace of $\mathcal{E}_T$ and thus inherits the scalar product $(\cdot,\cdot)$.

Definition 1.3.10. The set of *restricted roots* of θ is

$$\Phi_\theta = \left\{\bar{\alpha} = \frac{\alpha - \alpha^{\theta}}{2} \in \mathcal{E}_\theta : \alpha \in \Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{g}}^{\theta}\right\}.$$

The following is well known (see, for example [Ric82b]).

Proposition 1.3.11. *The set Φ_θ is a (possibly non-reduced) root system in $\mathcal{E}_\theta$ with Weyl group the* little Weyl group $W_\theta = N_G(A)/Z_G(A) = N_{G_0^\theta}(A)/Z_{G_0^\theta}(A)$.

Simple roots for Φ_θ can be constructed from the simple roots $\Delta_{\mathfrak{g}}$ of $\Phi_{\mathfrak{g}}$. Indeed, these roots can be split into two groups

$$\Delta_{\mathfrak{g}} \cap (\Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{g}}^{\theta}) = \{\alpha_1, \dots, \alpha_m\}, \quad \Delta_{\mathfrak{g}} \cap \Phi_{\mathfrak{g}}^{\theta} = \{\beta_1, \dots, \beta_{r-m}\}.$$

Now, one can order (see [DCP83] for more details) the simple roots $\{\alpha_1, \dots, \alpha_m\}$ in such a way that the $\bar{\alpha}_i$ are mutually distinct for all $i \le l$ and for each $j > l$ there exists some $i \le l$ with $\bar{\alpha}_j = \bar{\alpha}_i$. Thus, we obtain the set of *restricted simple roots*

$$\Delta_\theta = \{\bar{\alpha}_1, \dots, \bar{\alpha}_l\}.$$

There is a natural inclusion of the root lattice $\mathcal{R}_\theta := \mathcal{R}(\Phi_\theta) = \mathbb{Z}\langle\Phi_\theta\rangle$ in the group of characters $X^*(A)$, and the cokernel of this inclusion is the group of characters of the subgroup

$$\{a \in A : \bar{\alpha}(a) = 1 \text{ for all } \bar{\alpha} \in \Phi_\theta\}.$$

This group can be easily shown to be equal to $F_\theta = A \cap G_\theta$, and thus, if we denote $A_{G_\theta} = A/F_\theta$, we get

$$\mathcal{R}_\theta = X^*(A_{G_\theta}).$$

Dually, we get $\mathcal{P}_\theta^\vee = \mathcal{P}^\vee(\Phi_\theta) = X_*(A_{G_\theta})$.

We describe now the weight lattice $\mathcal{P}_\theta = \mathcal{P}(\Phi_\theta)$ of Φ_θ. Let us denote by $\varpi_1, \dots, \varpi_l$ the corresponding fundamental dominant weights (that is, we want these weights to satisfy $\langle \bar{\alpha}_i^\vee, \varpi_j \rangle = \delta_{ij}$ for $i = 1, \dots, l$). When G is semisimple simply-connected, these ϖ_i can be determined in terms of the fundamental dominant weights of $\Phi_{\mathfrak{g}}$. We can partition these into two sets

$$\{\omega_1, \dots, \omega_m\}, \ \{\zeta_1, \dots, \zeta_{r-m}\}$$

with $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$, $\langle \beta_i^\vee, \zeta_j \rangle = \delta_{ij}$ and all the $\langle \alpha_i^\vee, \zeta_j \rangle = \langle \beta_i^\vee, \omega_j \rangle = 0$.

Since a root α and its image α^θ must have the same length, we can have three posible cases: (1) $\alpha^\theta = -\alpha$, (2) $\langle \alpha^\vee, \alpha^\theta \rangle = 0$, and (3) $\langle \alpha^\vee, \alpha^\theta \rangle = 1$. Note that if there is a root α of type (3) then $s_{\alpha^\theta}(\alpha) = \alpha - \alpha^\theta$ must be a root in $\Phi_{\mathfrak{g}}$, and this root restricts to itself, which is equal to $2\bar{\alpha}$. This implies that Φ_θ is nonreduced and that we can take the restricted simple roots Δ_θ coming just from simple roots of types (1) and (2).

Now, for any $\chi \in X^*(T)$ we have

$$\langle \bar{\alpha}^\vee, \chi - \chi^\theta \rangle = \begin{cases} \langle \alpha^\vee, \chi \rangle & \text{in case (1)} \\ \langle \alpha^\vee, \chi - \chi^\theta \rangle & \text{in case (2)} \\ 2\langle \alpha^\vee, \chi - \chi^\theta \rangle & \text{in case (3).} \end{cases}$$

As in [DCP83], it is easy to see that there exists some permutation $\sigma \in \mathfrak{S}_m$ of order 2 such that for each $i = 1, \dots, l$, we have that $\alpha_i^\theta + \alpha_{\sigma(i)} = \alpha_{\sigma(i)}^\theta + \alpha_i$ is a positive imaginary root. Moreover, we have $\omega_i^\theta = -\omega_{\sigma(i)}$. Then it is straightforward from the above that we get

$$\varpi_i = \begin{cases} 2\omega_i & \text{if } \alpha_i \text{ is of type (1)} \\ \omega_i + \omega_{\sigma(i)} & \text{if } \alpha_i \text{ is of type (2) and } i \neq \sigma(i) \\ \omega_i & \text{if } \alpha_i \text{ is of type (2) and } i = \sigma(i). \end{cases}$$

Note that if $\Phi_{\mathfrak{g}}^\theta = \emptyset$, then $i = \sigma(i)$ if and only if $\alpha_i^\theta = -\alpha_i$, and thus we do not have the third situation in the formula above.

Lemma 1.3.12. *When* G *is semisimple simply-connected, the weight lattice* $\mathcal{P}_\theta = \mathbb{Z}\langle \varpi_1, \dots, \varpi_l \rangle$ *is equal to the group of characters* $X^*(A_{G^\theta})$ *of the torus* $A_{G^\theta} = A/F^\theta$. *Dually, we get* $\mathcal{R}_\theta^\vee = \mathcal{R}^\vee(\Phi_\theta) = X_*(A_{G^\theta})$.

Proof. The lattice $X^*(A_{G^\theta})$ can be identified with $2X^*(A) = \{\chi - \chi^\theta : \chi \in X^*(T)\}$. It is clear from the above that for every $\chi \in X^*(T)$ and $\bar{\alpha} \in \Phi_\theta$, the number $\langle \bar{\alpha}^\vee, \chi - \chi^\theta \rangle$ is an integer. On the other hand, $2\omega_i$ and $\omega_i + \omega_{\sigma(i)}$ are clearly of the form $\chi - \chi^\theta$, while clearly ω_i restricts to 1 on $F^\theta = A \cap G^\theta$ if $\omega_i^\theta = -\omega_i$. □

To finish the section, note that

$$\mathcal{P}_{\theta,+} := \mathcal{P}_+(\Delta_\theta) = \mathcal{P}_\theta \cap C^+_{\Delta_\theta} = \mathcal{P}_\theta \cap C^+_\Delta = X^*(A_{G^\theta}) \cap C^+_\Delta =: X^*(A_{G^\theta})_+.$$

By an analogous argument, we get $\mathcal{P}^\vee_{\theta,+} = X_*(A_{G_\theta}) \cap C^+_\Delta = X_*(A_{G_\theta})_+$.

Examples

Example 1.3.13 (The diagonal case)**.** In the diagonal case from Example 1.2.5, a maximal Θ-split torus is given by $A = \left\{(t^{-1}, t) : t \in T\right\} \subset T \times T$, with character lattice $X^*(A) = \{(-\chi, \chi) : \chi \in X^*(T)\}$. The restricted roots are then identified with $\Phi_{\mathfrak{g}}$.

Example 1.3.14 ($SL_n(\mathbb{C}), n > 2$)**.** Recall the classification of the involutions of $SL_n(\mathbb{C})$ given in Example 1.2.6. We determine now the restricted root systems associated to each of these involutions. We begin by considering the inner involutions $\theta_{p,q}(g) = I_{p,q} g I_{p,q}$. Under a change of basis, we can write these involutions as $\theta_{p,q}(g) = I'_{p,q} g I'_{p,q}$, for

$$I'_{p,q} = \begin{pmatrix} 0 & 0 & \tilde{I}_p \\ 0 & I_{q-p} & 0 \\ \tilde{I}_p & 0 & 0 \end{pmatrix},$$

where $\tilde{I}_p$ is the $p \times p$ matrix

$$\tilde{I}_p = \begin{pmatrix} 0 & \dots & 1 \\ 0 & \iddots & 0 \\ 1 & \dots & 0 \end{pmatrix}.$$

Now it is clear that the elements of the form

$$a = \operatorname{diag}(a_1, \dots, a_p, 1, \dots, 1, a_p^{-1}, \dots, a_1^{-1})$$

form a maximal $\theta_{p,q}$-split torus A. This torus sits inside the standard maximal torus T of $SL_n(\mathbb{C})$, which is θ-stable. The corresponding simple roots are $\Delta_{\mathfrak{sl}_n} = \{\alpha_1, \dots, \alpha_{n-1}\}$, with

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1},$$

for $\varepsilon_i(\operatorname{diag}(a_1, \dots, a_n)) = a_i$. The imaginary roots are those of the form $\varepsilon_i - \varepsilon_j$, for $p < i \neq j \leq q$; thus, $\theta_{p,q}$ is quasisplit if and only if $q = p$ or $q = p + 1$. The restricted root system is

$$\Phi_\theta = \left\{\pm\bar{\varepsilon}_i \pm \bar{\varepsilon}_j, \pm 2\bar{\varepsilon}_i, \pm\bar{\varepsilon}_i : 1 \leq i \neq j \leq p\right\},$$

when $p \neq n/2$, and

$$\Phi_\theta = \left\{\pm\bar{\varepsilon}_i \pm \bar{\varepsilon}_j, \pm 2\bar{\varepsilon}_i : 1 \leq i \neq j \leq p\right\},$$

when $p = n/2$. Therefore, the restricted root system is nonreduced, of type BC_p if $p \neq n/2$, and reduced of type C_p if $p = n/2$.

For the outer involution θ_0, the standard torus is maximal θ_0-split and $\Phi_\theta = \Phi_{\mathfrak{sl}_n}$. On the other hand, if $n = 2m$ is even, we can choose a basis such that $\theta_1(g) = I_{m,m}(g^{\tilde{T}})^{-1} I_{m,m}$, where $\tilde{T}$ denotes transposition with respect to the secondary diagonal. It is now easy to see that the elements of the form

$$a = \operatorname{diag}(a_1, a_2, \dots, a_2, a_1),$$

with $a_1 a_2 \cdots a_m = 1$ form a maximal θ-split torus A. The restricted root system now is

$$\Phi_\theta = \left\{\bar{\varepsilon}_i - \bar{\varepsilon}_j : 1 \leq i \neq j \leq m\right\},$$

so it has type A_{m-1}. Table 1.1 summarizes this example and Example 1.2.6.

Table 1.1: Involutions of $SL_n(\mathbb{C})$, for $n > 2$. Notation for real forms following Helgason [Hel01].

	Involution	G^θ	Real form	Split or quasisplit?	Φ_θ
Inner	$\theta_{p,q}(g) = I_{p,q} g I_{p,q}$	$S(GL_p(\mathbb{C}) \times GL_q(\mathbb{C}))$	$SU(p,q)$	Quasisplit iff $q = p$ or $q = p+1$	BC_p if $p \neq q$ C_p if $p = q$
Outer	$\theta_0(g) = (g^T)^{-1}$	$SO_n(\mathbb{C})$	$SL_n(\mathbb{R})$	Split	A_{n-1}
	$\theta_1(g) = J_m (g^T)^{-1} J_m^{-1}$ (only if $n = 2m$)	$Sp_{2m}(\mathbb{C})$	$SU^*(n)$	No	A_{m-1}

1.4 SYMMETRIC VARIETIES AND THEIR EMBEDDINGS

Symmetric pairs and symmetric varieties

Let G be a reductive algebraic group over $\mathbb{C}$.

Definition 1.4.1. Any closed subgroup $H \subset G$ of G with

$$G_0^\theta \subset H \subset G_\theta$$

for some involution $\theta \in \mathrm{Aut}_2(G)$ is called a *symmetric subgroup* (associated to θ). A pair (G, H) where G is a reductive group and $H \subset G$ is a symmetric subgroup is called a *symmetric pair*, while the corresponding algebraic homogeneous space G/H is called a *symmetric variety*.

An *embedding of a symmetric variety* or *symmetric embedding* is a normal G-variety Σ with a G-equivariant open embedding $O_\Sigma \hookrightarrow \Sigma$, where $O_\Sigma = G/H$ is a symmetric variety.

Remark 1.4.2. Note that the Lie algebra of any symmetric subgroup associated to an involution θ is equal to $\mathfrak{g}^\theta$.

If G is a reductive group, we can decompose it as $G = G' Z_G^0$, for G' the derived group and Z_G^0 the connected centre of G, which is a torus. Now, any involution $\theta \in \mathrm{Aut}_2(G)$ preserves the centre and its connected components, so θ' acts on G' as an involution $\theta' = \theta|_{G'} \in \mathrm{Aut}_2(G')$. Moreover, there exists a torus Z such that any symmetric variety G/H associated to θ is isomorphis to one associated to the involution $(\theta'(g), z) \mapsto (g, z^{-1})$ on $G' \times Z$.

Thus, from now on we will assume that any symmetric variety is of the form G_Z/H_Z where G_Z is a reductive group $G_Z = G \times Z$, with G semisimple and Z a torus, and that G_Z/H_Z is associated to the involution $(g, z) \mapsto (\theta(g), z^{-1})$ for $\theta \in \mathrm{Aut}_2(G)$. The group $H = H_Z \cap (G \times \{1\})$ is a symmetric subgroup of G associated to θ. We denote $(G_Z/H_Z)' = G/H$ and call it the *semisimple part* of G_Z/H_Z.

Affine spherical varieties

Let Σ be a normal G-variety. We denote by $\mathbb{C}[\Sigma]$ its ring of regular functions and by $\mathbb{C}(\Sigma)$ its function field. For any subgroup $H \subset G$ we denote by $\mathbb{C}[\Sigma]^H$

(respectively $\mathbb{C}(\Sigma)^H$) the corresponding ring (resp. field) of H-invariant functions; that is, of functions f with

$$f(h \cdot x) = f(x) \text{ for any } h \in H \text{ and } x \in \Sigma.$$

We denote by $\mathbb{C}[\Sigma]^{(H)}$ (respectively $\mathbb{C}(\Sigma)^{(H)}$) the corresponding ring (resp. field) of H-*semiinvariants* of Σ; these are the functions f such that there exists a character $\chi_f \in X^*(H)$ with

$$f(h \cdot x) = h^{\chi_f} f(x) \text{ for any } h \in H.$$

Let $B \subset G$ be a Borel subgroup. The characters that arise from the B-semiinvariants of Σ are called the *weights* of Σ. More precisely, we define the *weight lattice* and the *weight semigroup* of Σ to be, respectively

$$P(\Sigma) = \left\{\chi \in X^*(B) : \exists f \in \mathbb{C}(\Sigma)^{(B)}, \chi_f = \chi\right\},$$
$$P_+(\Sigma) = \left\{\chi \in X^*(B) : \exists f \in \mathbb{C}[\Sigma]^{(B)}, \chi_f = \chi\right\}.$$

The rank of the weight lattice $P(\Sigma)$ is called the *rank* of Σ.

When Σ is affine, $P(\Sigma)$ can be determined in terms of $P_+(\Sigma)$. Indeed, we have the following.

Proposition 1.4.3. *Suppose that Σ is an affine G-variety. Then every element $f \in \mathbb{C}(\Sigma)^{(B)}$ can be written as $f = f_1/f_2$, for $f_1, f_2 \in \mathbb{C}[\Sigma]^{(B)}$.*

Proof. Start by writing $f = f'_1/f'_2$ for $f'_1, f'_2 \in \mathbb{C}[\Sigma]$ not necessarily B-semiinvariant. Consider now the subspace of $\mathbb{C}[\Sigma]$ generated by the B-orbit $B \cdot f'_2$. This space is finite dimensional and B-stable and thus, by the Lie–Kolchin theorem, it must contain a B-semiinvariant f_2. Write $f_2 = \sum_i \xi_i(b_i \cdot f'_2)$, with $\xi_i \in \mathbb{C}$ and $b_i \in B$ and denote $f_1 = a \sum_i \xi_i(b_i \cdot f'_1)$, for

$$a = \frac{\sum_i \xi_i}{\sum_i \xi_i \chi_f(b_i)}.$$

Then for all i we have

$$\frac{b_i \cdot f'_1}{b_i \cdot f'_2} = b_i \cdot f = \chi_f(b_i) \cdot f$$

and thus

$$\frac{f_1}{f_2} = f,$$

so $f_1 = f f_2$ is also B-semiinvariant. □

Corollary 1.4.4. *If Σ is an affine G-variety, we have*

$$P(\Sigma) = \mathbb{Z}P_+(\Sigma).$$

Definition 1.4.5. A normal G-variety Σ is *spherical* if there exists a Borel subgroup $B \subset G$ and an open B-orbit in Σ.

In general, there is a natural short exact sequence

$$0 \longrightarrow \mathbb{C}(\Sigma)^B \longrightarrow \mathbb{C}(\Sigma)^{(B)} \longrightarrow P(\Sigma) \longrightarrow 0.$$

Now, suppose that Σ is spherical, choose B with an open B-orbit in Σ, and take x_0 a point in that orbit. The orbit Bx_0 is birational to Σ and therefore we have

$$\mathbb{C}(\Sigma)^B = \mathbb{C}(Bx_0)^B = \mathbb{C}^*$$

Note that, in fact, this property is equivalent to Σ being a spherical variety. Moreover, if we denote by $H \subset G$ the stabilizer of x_0, we have $Bx_0 \cong B/H$ and thus we get $\mathbb{C}(\Sigma)^{(B)} = \mathbb{C}(Bx_0)^{(B)} = \mathbb{C}(B/H)^{(B)}$ and

$$P(\Sigma) = X^*(B/(B \cap H)).$$

By restricting characters to a maximal torus $T \subset B$, we conclude the following.

Proposition 1.4.6. *Let Σ be a spherical variety with open B-orbit $Bx_0 \cong B/H$, $T \subset B$ a maximal torus and put $T_\Sigma = T/(T \cap H)$. Then, there is a canonical bijection*

$$P(\Sigma) = X^*(T_\Sigma).$$

Let us suppose now that Σ is an affine G-variety. Then, the coordinate ring $\mathbb{C}[\Sigma]$ is naturally a G-module, and as such it decomposes as a sum of irreducible representations of G

$$\mathbb{C}[\Sigma] = \bigoplus_{\chi \in X^*(T)_+} \mathbb{C}[\Sigma]_\chi \cong \bigoplus_{\chi \in X^*(T)_+} m_\chi V_\chi,$$

for $m_\chi = m_{V_\chi}(\Sigma)$, and V_χ the irreducible representation of highest weight χ. We write $\mathbb{C}[\Sigma]_\chi$ for the component isomorphic to $m_\chi V_\chi$. This allows us to identify

$$P_+(\Sigma) = \left\{\chi \in X^*(T)_+ : m_\chi \geq 1\right\},$$

for $T \subset G$ any maximal torus. In particular, we have a natural inclusion $P_+(\Sigma) \subset X^*(T)$. If $B \supset T$ is a Borel subgroup of G, then the multiplicities m_χ can be computed as

$$m_\chi = \dim \mathbb{C}[\Sigma]_\chi^{(B)},$$

where $\mathbb{C}[\Sigma]_\chi^{(B)}$ denotes the subspace of B-semiivariants with weight χ. Also, if $\Sigma = G/H$ is a homogeneous space, the multiplicities can also be obtained as

$$m_\chi = \dim(V_\chi^*)^H.$$

Proposition 1.4.7. *An affine G-variety is spherical if and only if its coordinate ring decomposes as*

$$\mathbb{C}[\Sigma] = \bigoplus_{\chi \in P_+(\Sigma)} V_\chi.$$

Proof. Recall that, if Σ is affine, every element of $\mathbb{C}(\Sigma)^{(B)}$ can be written as a quotient f_1/f_2, for $f_1, f_2 \in \mathbb{C}[\Sigma]^{(B)}$. Now, $f_1/f_2 \in \mathbb{C}(\Sigma)^B$ if and only if $\chi_{f_1} = \chi_{f_2}$. The variety Σ is spherical if and only if $\mathbb{C}(\Sigma)^B = \mathbb{C}^*$, and thus if and only if any two $f_1, f_2 \in \mathbb{C}[\Sigma]^{(B)}$ with the same weight χ are proportional; that is, if and only if $m_\chi \leq 1$ for every χ. □

When Σ is an affine spherical variety, the weight semigroup $P_+(\Sigma)$ can be recovered as the lattice points on the cone generated by it.

Proposition 1.4.8. *If Σ is an affine spherical variety, then*

$$P_+(\Sigma) = \mathbb{Q}_+ P_+(\Sigma) \cap \mathbb{Z} P_+(\Sigma).$$

Proof. Recall that $\mathbb{Z}P_+(\Sigma) = P(\Sigma)$ so for any $\chi \in \mathbb{Z}P_+(\Sigma)$ there exists a function $f \in \mathbb{C}(\Sigma)^{(B)}$ with $\chi_f = \chi$. Now, $\chi \in \mathbb{Q}_+P_+(\Sigma)$ if and only if there exists some natural number $n \in \mathbb{Z}_+$ such that $n\chi \in P_+(\Sigma)$. Since f has weight χ, the power f^n has weight $n\chi$ and thus $f^n \in \mathbb{C}[\Sigma]$. But, since Σ is normal, $f \in \mathbb{C}[\Sigma]$ and $\chi \in P_+(\Sigma)$. □

The weights of a symmetric variety

Proposition 1.4.9. *Symmetric varieties and their embeddings are spherical.*

Proof. By definition, a symmetric embedding is a normal G-variety Σ with an open embedding of a symmetric variety G/H. Thus, it suffices to prove that symmetric varieties are spherical. Suppose that G/H is a symmetric variety associated to an involution $\theta \in \mathrm{Aut}_2(G)$. Since $G_0^\theta \subset H$, there is a natural projection $G/G_0^\theta \to G/H$, so it suffices to prove that G/G_0^θ is spherical, since the projection of an open B-orbit in G/G_0^θ gives an open B-orbit in G/H. Now, G/G_0^θ is spherical as a consequence of the Iwasawa decomposition. Indeed, recall that the Iwasawa decomposition says that there is an open embedding $G_0^\theta A P^u \subset G$, where P^u is the unipotent radical of a minimal θ-split parabolic subgroup $P \subset G$. This induces an open G-equivariant embedding $AP^u \hookrightarrow G/G_0^\theta$ and, if we choose any Borel subgroup $B \subset P$, we have $AP^u \subset B$ and therefore we get an open G-equivariant embedding $B \hookrightarrow G/G_0^\theta$. □

A first consequence of symmetric varieties being spherical is that, if Σ is a symmetric embedding with $O_\Sigma = G/H$ a symmetric variety associated to an involution $\theta \in \mathrm{Aut}_2(G)$, then

$$P(\Sigma) = X^*(T_H),$$

where $T_H = T/(T \cap H)$ for any θ-stable maximal torus $T \subset G$. Note that if $A \subset G$ is a maximal θ-split torus contained in T, then we also have

$$T_H = A_H = A/(A \cap H).$$

In particular, we have

$$P(G/G_\theta) = X^*(A_{G_\theta}) = \mathcal{R}_\theta.$$

Moreover, we have

$$P(G/G^\theta) = X^*(A_{G^\theta})$$

which, if G is semisimple simply-connected, is equal to the weight lattice $\mathcal{P}_\theta$.

In general, one can recover the weight semigroup of a symmetric variety G/H as the dominant weights of the corresponding torus A_H. More precisely, we have the following.

Proposition 1.4.10. *For any symmetric variety* G/H*, we have*

$$\mathbb{Q}_+P_+(G/H) = C^+_{\Delta_\theta}.$$

Therefore,

$$P_+(G/H) = X^*(A_H) \cap C^+_{\Delta_\theta} = X^*(A_H) \cap C^+_{\Delta} =: X^*(A_H)_+.$$

Proof. Note that $P_+(G/H)$ is clearly contained in C^+_Δ and in $\mathcal{E}_\theta$ (since $X^*(A_H) \subset \mathcal{E}_\theta$), and therefore it is contained in $\mathcal{E}_\theta \cap C^+_\Delta = C^+_{\Delta_\theta}$.

For the other inclusion, it suffices to show that for any $\chi \in X^*(T) \cap C^+_{\Delta_\theta}$ we have $2\chi \in P_+(G/H)$. We show this by proving that $(V^*_{2\chi})^H \neq \{0\}$. Note that an element $\chi \in X^*(T) \cap C^+_{\Delta_\theta}$ is just a dominant character of T with $\chi^\theta = -\chi$. We now need the following.

Lemma 1.4.11. *If* χ *is a dominant character of* T *with* $\chi^\theta = -\chi$*, then there is an isomorphism* $V^\theta_\chi \to V^*_\chi$*, where* V^θ_χ *is the space* V_χ *endowed with the* θ*-twisted* G*-action* $g \cdot_\theta v = \theta(g) \cdot v$*.*

The isomorphism $V^\theta_\chi \to V^*_\chi$ can be understood as a θ-linear isomorphism $V_\chi \to V^*_\chi$. Under this isomorphism, the highest weight vector $v_\chi \in V_\chi$ is mapped into a lowest weight vector in V^*_χ. We can now describe a canonical way to pick a θ-linear isomorphism. In V_χ the line $\mathbb{C}v_\chi$ has a unique T-stable complement $\bar{V}_\chi$, and we define $v^\chi \in V_\chi$ by $\langle v^\chi, v_\chi\rangle = 1$, $\langle v^\chi, \bar{V}_\chi\rangle = 0$. This v^χ is a lowest weight vector, and thus we can define $\omega : V^*_\chi \to V_\chi$ to be the unique θ-linear isomorphism with $\omega(v^\chi) = v_\chi$.

Complete now v_χ to a basis $\{v_\chi, v_1, v_2, \dots, v_m\}$ of weight vectors and consider the dual basis $\{v^\chi, v^1, v^2, \dots, v^m\}$. We have $\omega(v^\chi) = v_\chi$ and if χ_i is the weight of v_i, the weight of v^i is $-\chi_i$, so $-\chi_i^\theta$ is the weight of $w_i = \omega(v^i)$. Now, under the isomorphism $\mathrm{Hom}(V^*_\chi, V_\chi) \cong V_\chi \otimes V_\chi$, the homomorphism ω is identified with

$$\omega = v_\chi \otimes v_\chi + \sum_{i=1}^m w_i \otimes v_i,$$

where $v_\chi \otimes v_\chi$ has weight 2χ and $w_i \otimes v_i$ has weight $\chi_i - \chi_i^\theta$.

Since ω is θ-linear, it is an H-equivariant isomorphism, and thus it is naturally an element of $(V_\chi \otimes V_\chi)^H$. Taking the image under the canonical G-equivariant projection $V_\chi \otimes V_\chi \to V_{2\chi}$, we obtain a non-zero element $\bar{\omega} \in V^H_{2\chi}$. By the dual process, we obtain a non-zero element in $(V^*_{2\chi})^H$. ◻

We give now the proof of the Lemma.

Proof of Lemma 1.4.11. The G-module V^*_χ can be characterized as the irreducible representation of G with lowest weight $-\chi$. Now if $v_\chi \in V_\chi$ is a vector of weight χ and we consider $P \subset G$ the parabolic subgroup of G fixing the line through v_χ, then P is generated by a Borel subgroup B and the root subgroups relative to the negative roots $-\alpha$ for which $(\alpha, \chi) = 0$. Therefore $\theta(P)$ contains the root subgroups relative to the simple imaginary roots and their negatives, and also to the roots α^θ, for $\alpha \in \Phi^+_{\mathfrak{g}} \setminus \Phi^\theta_{\mathfrak{g}}$. Now, since θ maps positive roots in $\Phi^+_{\mathfrak{g}} \setminus \Phi^\theta_{\mathfrak{g}}$ to negative roots, $\theta(P)$ contains a Borel subgroup opposite to B. Clearly $v_\lambda \in V^\theta_\lambda$ is stabilized by $\theta(P)$ and therefore it is a minimal weight vector of weight $-\lambda$. ◻

The dual group

Sakellaridis and Venkatesh [SV17] describe a canonical way of associating a *dual group* $\check{G}_\Sigma$ to any spherical G-variety Σ, conjecturally with a natural homomorphism $\check{G}_\Sigma \to \check{G}$ to the Langlands dual group $\check{G}$ of G. Knop and Schalke [KS17] later proved that, indeed, $\check{G}_\Sigma$ is contained in $\check{G}$ up to isogeny. Their construction consists on taking the weight lattice $P(\Sigma)$ on Σ and a natural set of simple roots Δ_Σ associated to Σ, in such a way that $(P(\Sigma), \Delta_\Sigma, P(\Sigma)^\vee, \Delta_\Sigma^\vee)$ is a based root datum. The dual group $\check{G}_\Sigma$ is defined as the reductive group over $\mathbb{C}$ with based root datum equal to $(P(\Sigma)^\vee, \Delta_\Sigma^\vee, P(\Sigma), \Delta_\Sigma)$. The simple roots Δ_Σ are obtained by taking the generators of intersections of the weight lattice with the negative of the dual of the *central valuation cone*, and then applying some process of normalization, consisting in further intersecting the real span of each generator with the root lattice of G, as explained in [SV17, KS17].

If $\theta \in \mathrm{Aut}_2(G)$ is an involution, there is a natural reduced root system associated to θ, namely

$$\Phi_\theta^{\mathrm{red}} = \{\alpha \in \Phi_\theta : \alpha/2 \notin \Phi_\theta\},$$

the reduced version of the restricted root system obtained by taking the shortest roots. If A is the corresponding maximal θ-split torus, the resulting based root datum is $(X^*(A), \Delta_\theta^{\mathrm{red}}, X_*(A), (\Delta_\theta^{\mathrm{red}})^\vee)$. The group $\tilde{G}_\theta$ corresponding to this based root datum is the complexification of a *maximal* $\mathbb{R}$*-split subalgebra* $\tilde{\mathfrak{g}}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$ inside the Lie algebra $\mathfrak{g}_\mathbb{R}$ of the real form of G corresponding to θ. See [GPPNR18, Section 2.2] for more details.

In most cases $\Delta_{G/G^\theta} = \Delta_\theta^{\mathrm{red}}$, but in some cases there are normalization factors involved, that change some roots by multiples of them by factors of 1/2 or 2. In specific cases, we can find that $\Delta_\theta^{\mathrm{red}}$ is of type B_m or C_m, whereas Δ_{G/G^θ} is of type C_m or B_m, respectively. In Table 1.2, we recollect the different involutions of the simple Lie algebras with their fixed points, their associated real forms and root systems, and the Lie algebra of the corresponding dual groups. The reader can compare Table 1.2 with Table 1 in [GPPNR18] and with Table 1 in [Nad05]. For us, the most relevant properties of the dual group $\check{G}_{G/G^\theta}$ are the following:

1. $\check{G}_{G/G^\theta}$ contains $\check{A}_{G^\theta} = \mathrm{Spec}(\mathbb{C}[e^{X_*(A_{G^\theta})}])$ as a maximal torus,

2. the Weyl group of $\check{G}_{G/G^\theta}$ is the little Weyl group W_θ,

3. the dominant Weyl chamber of $\check{G}_{G/G^\theta}$ coincides with $X_*(A_{G^\theta})_+$.

The dual group $\check{G}_{G/G^\theta}$ of the symmetric variety G/G^θ was previously discovered by Nadler [Nad05] as a group $\check{G}_{G_\mathbb{R}}$ naturally associated to the corresponding real form $G_\mathbb{R}$, obtained as the Tannaka group of a certain neutral Tannakian category of perverse sheaves in the *real loop Grassmannian* of $G_\mathbb{R}$, thus giving a generalization of the geometric Satake correspondence. More generaly, by a similar procedure, Gaitsgory and Nadler [GN10] associated a reductive group $\check{G}_{\Sigma,GN}$ to any spherical G-variety Σ, with a natural inclusion $\check{G}_{\Sigma,GN} \subset \check{G}$. The equality of $\check{G}_{\Sigma,GN}$ and $\check{G}_\Sigma$ remains conjectural in the general case of a spherical variety [BZSV23, Page 75].

Table 1.2: Simple groups with their involutions, real forms and associated dual groups. Notation for types and real forms following Helgason [Hel01].

Type	$\mathfrak{g}$	$\check{\mathfrak{g}}$	$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{g}^\theta$	Δ_θ	Δ_θ^{red}	Δ_{G/G^θ}	$\check{\mathfrak{g}}_{G/G^\theta}$	Notes
AI	$\mathfrak{sl}_n$	$\mathfrak{sl}_n$	$\mathfrak{sl}_n(\mathbb{R})$	$\mathfrak{so}_n$	A_{n-1}	A_{n-1}	A_{n-1}	$\mathfrak{sl}_n$	split
AII	$\mathfrak{sl}_n$	$\mathfrak{sl}_n$	$\mathfrak{so}^*(2n)$	$\mathfrak{sp}_{2n}$	A_{n-1}	A_{n-1}	A_{n-1}	$\mathfrak{sl}_n$	
AIII	$\mathfrak{sl}_n$	$\mathfrak{sl}_n$	$\mathfrak{so}(p,q)$	$\mathfrak{s}(\mathfrak{gl}_p \oplus \mathfrak{gl}_q)$	BC_p	B_p	B_p	$\mathfrak{sp}_{2p}$	$p+q=n, p<q$ QS if $q=p+1$
AIV	$\mathfrak{sl}_{2n}$	$\mathfrak{sl}_{2n}$	$\mathfrak{su}(n,n)$	$\mathfrak{s}(\mathfrak{gl}_n \oplus \mathfrak{gl}_n)$	C_n	C_n	B_n	$\mathfrak{sp}_{2n}$	quasisplit $\Delta_\theta^{red} \neq \Delta_{G/G^\theta}$
BI	$\mathfrak{so}_{2n+1}$	$\mathfrak{sp}_{2n}$	$\mathfrak{so}(p,q+1)$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$	B_p	B_p	B_p	$\mathfrak{sp}_{2p}$	$p+q=2n, p<q$
BII	$\mathfrak{so}_{2n+1}$	$\mathfrak{sp}_{2n}$	$\mathfrak{so}(n,n+1)$	$\mathfrak{so}_n \oplus \mathfrak{so}_{n+1}$	B_n	B_n	B_n	$\mathfrak{sp}_{2n}$	split
CI	$\mathfrak{sp}_{2n}$	$\mathfrak{so}_{2n+1}$	$\mathfrak{sp}_{2n}(\mathbb{R})$	$\mathfrak{gl}_n$	C_n	C_n	C_n	$\mathfrak{so}_{2n+1}$	split
CII	$\mathfrak{sp}_{2n}$	$\mathfrak{so}_{2n+1}$	$\mathfrak{sp}(2p,2q)$	$\mathfrak{sp}_{2p} \oplus \mathfrak{sp}_{2q}$	BC_p	B_p	B_p	$\mathfrak{sp}_{2p}$	$p+q=n, p<q$
	$\mathfrak{sp}_{4n}$	$\mathfrak{so}_{4n+1}$	$\mathfrak{sp}(2n,2n)$	$\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2n}$	C_n	C_n	B_n	$\mathfrak{sp}_{2n}$	$\Delta_\theta^{red} \neq \Delta_{G/G^\theta}$
DI	$\mathfrak{so}_{2n}$	$\mathfrak{so}_{2n}$	$\mathfrak{so}(n,n)$	$\mathfrak{so}_n \oplus \mathfrak{so}_n$	D_n	D_n	D_n	$\mathfrak{so}_{2n}$	split
DII	$\mathfrak{so}_{2n}$	$\mathfrak{so}_{2n}$	$\mathfrak{so}(p,q)$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$	B_p	B_p	C_p	$\mathfrak{so}_{2p+1}$	$p+q=2n, p<q$ QS if $q=p+2$ $\Delta_\theta^{red} \neq \Delta_{G/G^\theta}$
DIII	$\mathfrak{so}_{2n}$	$\mathfrak{so}_{2n}$	$\mathfrak{so}^*(2n)$	$\mathfrak{gl}_n$	$BC_{(n-1)/2}$	$B_{(n-1)/2}$	$B_{(n-1)/2}$	$\mathfrak{sp}_{n-1}$	n odd
	$\mathfrak{so}_{2n}$	$\mathfrak{so}_{2n}$	$\mathfrak{so}^*(2n)$	$\mathfrak{gl}_n$	$C_{n/2}$	$C_{n/2}$	$B_{n/2}$	$\mathfrak{sp}_n$	n even $\Delta_\theta^{red} \neq \Delta_{G/G^\theta}$
EI	$\mathfrak{e}_6$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}_8$	E_6	E_6	E_6	$\mathfrak{e}_6$	split
EII	$\mathfrak{e}_6$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(2)}$	$\mathfrak{sl}_6 \oplus \mathfrak{sl}_2$	F_4	F_4	F_4	$\mathfrak{f}_4$	quasisplit
EIII	$\mathfrak{e}_6$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}_{11} \oplus \mathbb{C}$	BC_2	B_2	C_2	$\mathfrak{so}_5$	$\Delta_\theta^{red} \neq \Delta_{G/G^\theta}$
EIV	$\mathfrak{e}_6$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_4$	A_2	A_2	A_2	$\mathfrak{sl}_3$	
EV	$\mathfrak{e}_7$	$\mathfrak{e}_7$	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}_8$	E_7	E_7	E_7	$\mathfrak{e}_7$	split
EVI	$\mathfrak{e}_7$	$\mathfrak{e}_7$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}_6 \oplus \mathfrak{sl}_2$	F_4	F_4	F_4	$\mathfrak{f}_4$	
EVII	$\mathfrak{e}_7$	$\mathfrak{e}_7$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6 \oplus \mathbb{C}$	C_3	C_3	B_3	$\mathfrak{sp}_6$	$\Delta_\theta^{red} \neq \Delta_{G/G^\theta}$
EVIII	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}_{16}$	E_8	E_8	E_8	$\mathfrak{e}_8$	split
EIX	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_7 \oplus \mathfrak{sl}_2$	F_4	F_4	F_4	$\mathfrak{f}_4$	
FI	$\mathfrak{f}_4$	$\mathfrak{f}_4$	$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}_6 \oplus \mathfrak{sl}_2$	F_4	F_4	F_4	$\mathfrak{f}_4$	split
FII	$\mathfrak{f}_4$	$\mathfrak{f}_4$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}_9$	BC_1	A_1	A_1	$\mathfrak{sl}_2$	
G	$\mathfrak{g}_2$	$\mathfrak{g}_2$	$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$	G_2	G_2	G_2	$\mathfrak{g}_2$	split

1.5 THE WONDERFUL COMPACTIFICATION

Wonderful varieties

As in the previous section, we let G be a reductive algebraic group over $\mathbb{C}$.

Definition 1.5.1. Let G/H be a spherical homogeneous space and Σ a G-variety with a G-equivariant open and dense embedding $G/H \hookrightarrow \Sigma$. We say that Σ is *wonderful* if

1. Σ is smooth and projective.

2. $\Sigma \setminus (G/H)$ is a divisor with normal crossings, i.e., its components $D_1, \dots, D_l$ are smooth and intersect transversally.

3. The closures of the G-orbits of Σ are given by the intersections $D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_k}$, for $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq l$.

It follows from the theory of spherical varieties that, given a spherical homogeneous space G/H, a wonderful G-variety Σ with a G-equivariant open and dense embedding $G/H \hookrightarrow \Sigma$, if it exists, is unique. In the case that it exists, we call it the *wonderful compactification* of G/H. Moreover, one can obtain necessary and sufficient conditions for the existence of such a wonderful compactification. For the purpose of this document, we will just restrict ourselves to explain the proof given by De Concini and Procesi [DCP83] of the existence of the wonderful compactification of the symmetric variety G/G_θ, for $\theta \in \mathrm{Aut}_2(G)$. We refer the reader to [Tim11, Section 30] for a more general treatment of wonderful varieties.

The De Concini–Procesi embedding

Let $\theta \in \mathrm{Aut}_2(G)$ be an involution and choose $A \subset G$ a maximal θ-split torus and $T \subset G$ a maximal torus containing A. We also fix $B \subset G$ a Borel subgroup of G contained in a minimal θ-split parabolic and containing T, and denote by B^- the opposite Borel subgroup of B; that is $B \cap B^- = T$.

Let $\chi \in X^*(T)$ be a character of T lying in the interior of the dominant Weyl chamber and such that $\chi^\theta = -\chi$. Recall from the proof of Proposition 1.4.10 that there exists a canonical θ-linear element $\omega \in V_\chi \otimes V_\chi$, and that it projects to a non-zero element $\bar{\omega} \in V_{2\chi}$. Consider now the projective space $\mathbb{P}_{2\chi} = \mathbb{P}(V_{2\chi})$ and the corresponding point $[\bar{\omega}] \in \mathbb{P}_{2\chi}$. The action of G on $V_{2\chi}$ extends to an action on $\mathbb{P}_{2\chi}$.

Lemma 1.5.2. *The stabilizer of* $[\bar{\omega}] \in \mathbb{P}_{2\chi}$ *under the action of* G *is equal to* G_θ.

Proof. First, note that since ω is θ-linear we have

$$g \cdot \omega = g\omega g^{-1} = g\theta(g)^{-1}\omega.$$

Now, g stabilizes $[\bar{\omega}]$ if and only if $g \cdot \omega = a\omega$, for $a \in \mathbb{C}$ and thus if and only if $g\theta(g)^{-1}$ acts as a scalar on V_χ. But, since V_χ is irreducible, $g\theta(g)^{-1}$ acts on it as a scalar if and only if it belongs to the centre Z_G. □

Therefore, the G-orbit $G \cdot [\bar{\omega}]$ is canonically isomorphic to the symmetric variety G/G_θ. We denote by $\overline{G/G_\theta} = \overline{G \cdot [\bar{\omega}]} \subset \mathbb{P}_{2\chi}$ the closure of the orbit $G \cdot [\bar{\omega}]$ inside the projective space $\mathbb{P}_{2\chi}$, which is endowed with a natural embedding

$$G/G_\theta \xrightarrow{\sim} G \cdot [\bar{\omega}] \hookrightarrow \overline{G/G_\theta}.$$

Moreover, note that since $\overline{G/G_\theta}$ is closed in $\mathbb{P}_{2\chi}$ and G-stable, it contains the unique closed orbit of G acting on $\mathbb{P}_{2\chi}$, namely the orbit of the highest weight vector $v_{2\chi}$. We denote this orbit by $\Xi \subset \overline{G/G_\theta}$. A G-variety with a unique closed G-orbit is said to be *simple*.

Our purpose in this section is to give a sketch of the proof of the following [DCP83].

Theorem 1.5.3 (De Concini–Procesi)**.** *The projective variety $\overline{G/G_\theta}$ is the wonderful compactification of G/G_θ.*

The toric variety

In $V_{2\chi}$ the line $\mathbb{C}v_{2\chi}$ has a unique T-stable complement $\bar{V}_{2\chi}$ and we can consider the open affine subset $v_{2\chi} \oplus \bar{V}_{2\chi} \subset \mathbb{P}_{2\chi}$. Recall from the proof of 1.4.10 that we can decompose

$$\omega = v_\chi \otimes v_\chi + \sum_{i=1}^{m} w_i \otimes v_i,$$

and that the $w_i \otimes v_i$ had weights of the form $\chi_i - \chi_i^\theta$. This implies that we can decompose $\bar{\omega}$ in the form

$$\bar{\omega} = v_{2\chi} + \sum_{i=1}^{m} z_i.$$

Lemma 1.5.4. *The z_i above can be taken to have weights $2(\chi - \bar{\alpha}_i)$.*

Proof. Since $\bar{\omega}$ is θ-linear, for any $x \in \mathfrak{g}^\theta$ we have $e^x\bar{\omega} = 0$. In particular, if $\bar{\alpha} \in \Delta_\theta$ is the restriction of a simple root α, then for any $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ we have

$$e^{x_{-\alpha} + x_{-\alpha}^\theta}\bar{\omega} = 0.$$

But

$$e^{x_{-\alpha} + x_{-\alpha}^\theta} v_{2\chi} = e^{x_{-\alpha}} v_{2\chi}$$

and $e^{x_{-\alpha}} v_{2\chi}$ is a non-zero weight vector of weight $2\chi - \alpha$. Therefore, we can take z_i to satisfy $e^{x_{-\alpha}^\theta} z_i = -e^{x_{-\alpha}} v_{2\chi}$ and thus z_i has weight $2(\chi - \bar{\alpha})$. □

Lemma 1.5.5. *The closure in $v_{2\chi} \oplus \bar{V}_{2\chi}$ of the A-orbit $A \cdot [\bar{\omega}]$ is isomorphic to the A-toric variety*

$$\mathbb{A} = \operatorname{Spec}\left(\bigoplus_{\bar{\alpha} \in \Delta_\theta} \mathbb{C}[e^{-2\bar{\alpha}}]\right).$$

In other words, $\mathbb{A}$ is identified with the l-dimensional affine space $\mathbb{A}^l$ with the embedding $A \hookrightarrow \mathbb{A}^l$ given explicitly as $a \mapsto (a^{-2\bar{\alpha}_1}, \dots, a^{-2\bar{\alpha}_l})$ so that A is identified with the open set in $\mathbb{A}^l$ where all coordinates are non-zero.

Proof. Indeed we just have to apply an element $a \in A$ to $\bar{\omega}$. We obtain

$$a\bar{\omega} = a^{2\chi} v_{2\chi} \sum_{i=1}^{m} a^{2(\chi - \bar{\alpha}_i)} z_i.$$

In $\mathbb{P}_{2\chi}$ this defines the point with homogeneous coordinates

$$(1 : a^{-2\bar{\alpha}_1} : a^{-2\bar{\alpha}_2} : \cdots : a^{-2\bar{\alpha}_l}),$$

as we wanted to show. □

The local structure theorem

Consider the open subset $\mathrm{U} \subset \overline{\mathrm{G}/\mathrm{G}_\theta}$ defined as the intersection

$$\mathrm{U} = \overline{\mathrm{G}/\mathrm{G}_\theta} \cap (\nu_{2\chi} \oplus \bar{\mathrm{V}}_{2\chi})$$

inside of $\mathbb{P}_{2\chi}$. Note that U is B^--stable and that it contains $[\bar{\omega}]$ and $\mathbb{A}$. Therefore, $\nu_{2\chi} \in \mathrm{U}$ and thus U intersects the unique closed orbit Ξ. The following easy lemma allows us to obtain a lot of information about $\overline{\mathrm{G}/\mathrm{G}_\theta}$ from the study of Ξ.

Lemma 1.5.6. *If* Σ *is a* G*-variety with a unique closed* G*-orbit* Ξ *and* $\mathrm{U} \subset \Sigma$ *is an open subset with* $\Xi \cap \mathrm{U} \neq \varnothing$*, then*

$$\Sigma = \bigcup_{g \in \mathrm{G}} g \cdot \mathrm{U}.$$

The *local structure theorem* gives a nice description of the open subset U. Let P^u be the unipotent radical of B^-, which is the unipotent group with Lie algebra

$$\mathfrak{p}^u = \bigoplus_{\alpha \in \Phi_{\mathfrak{g}}^- \setminus \Phi_{\mathfrak{g}}^\theta} \mathfrak{g}_\alpha.$$

Since U is B^--stable, the unipotent group P^u acts on U and there is a well defined map

$$\begin{aligned} \mathrm{P}^u \times \mathbb{A} &\longrightarrow \mathrm{U} \\ (u, x) &\longmapsto u \cdot x. \end{aligned}$$

Theorem 1.5.7 (Local structure theorem)**.** *The above map* $\mathrm{P}^u \times \mathbb{A} \to \mathrm{U}$ *is an isomorphism.*

Remark 1.5.8. When restricting to $\mathrm{G}/\mathrm{G}_\theta$, the local structure theorem simply yields the Iwasawa decomposition $\mathrm{P}^u \times \mathrm{A} \hookrightarrow \mathrm{G}/\mathrm{G}_\theta$.

In order to make our exposition shorter, we omit the proof of the local structure theorem and refer the reader to [DCP83, Sections 2.3-2.7].

The boundary

The last ingredient for the proof of the theorem of De Concini–Procesi is the description of the boundary divisor $\overline{\mathrm{G}/\mathrm{G}_\theta} \setminus (\mathrm{G}/\mathrm{G}_\theta)$. A first result is the following.

Proposition 1.5.9. *The intersection between* $\mathrm{G} \cdot [\bar{\omega}]$ *and* $\mathrm{P}^u \times \mathbb{A}$ *is the open set where the last* l *coordinates are non-zero.*

Again, to simplify the exposition, we omit the proof of this proposition and refer the reader to [DCP83, Section 2.8].

Let us denote by $\mathrm{H}_i \subset \mathrm{P}^u \times \mathbb{A}$ the hyperplane formed by the points such that the i-th coordinate in $\mathbb{A}$ vanishes. From the description of the action of A in $\mathbb{A}$ it follows that two points in U lie in the same $\mathrm{P}^u \times \mathrm{A}$-orbit if and only if they are contained in exactly the same set of hyperplanes H_i.

Suppose now that D is an irreducible component of the boundary divisor $\overline{G/G_\theta} \setminus (G/G_\theta)$. Since G is assumed to be connected, necessarily D is G-stable. Therefore, the unique closed orbit Ξ is contained in D and thus $D \cap U$ is a component of $U \setminus G \cdot [\bar{\omega}]$. Therefore, there exists some $i \in \{1, \dots, l\}$ such that $D \cap U = H_i$. This implies that $D = \bar{H}_i$, so we denote it by $D_i = \bar{H}_i$.

Summing up, the components of the boundary divisor are of the form $D_i = \bar{H}_i$, for $i = 1, \dots, l$, and thus are smooth and intersect transversally. It is clear now that the orbits are of the form

$$D_{i_1} \cap \cdots \cap D_{i_k} \setminus \bigcup_{i \neq i_1, \dots, i_k} S_{i_1} \cap \cdots \cap S_{i_k} \cap S_i$$

and their closures are just $D_{i_1} \cap \cdots \cap D_{i_k}$. Moreover, it is also clear now that the unique closed orbit is

$$\Xi = \bigcap_{i=1}^{l} D_i.$$

We conclude then that $\overline{G/G_\theta}$ is indeed the wonderful compactification of G/G_θ.

1.6 VERY FLAT SYMMETRIC EMBEDDINGS

Abelianization

Let Σ be a simple affine symmetric embedding. Recall from our previous discussions that the corresponding symmetric variety O_Σ can be written in the form $O_\Sigma = G_Z/H_Z$, for G_Z a reductive group $G_Z = G \times Z$, with G semisimple and Z a torus and G_Z/H_Z associated to the involution $\vartheta : (g, z) \mapsto (\theta(g), z^{-1})$ for $\theta \in \mathrm{Aut}_2(G)$.

Consider the natural projections $\mathrm{pr}_1 : G_Z \to G$ and $\mathrm{pr}_2 : G_Z \to Z$. We can now define the tori

$$A_\Sigma = O_\Sigma/G = Z/\mathrm{pr}_2(H_Z),$$

and

$$Z_\Sigma = Z/(\mathrm{pr}_2(H_Z) \cap Z_2), \text{ for } Z_2 = \left\{z \in Z : z^2 = 1\right\}.$$

Naturally, we have a projection $Z_\Sigma \to A_\Sigma$ and an inclusion $X^*(A_\Sigma) \to X^*(Z_\Sigma)$.

Let $A \subset G$ be a maximal θ-split torus and $T \subset G$ a maximal torus containing it. Then, $A_Z = A \times Z$ is a maximal ϑ-split torus of G_Z and T_Z a maximal ϑ-stable torus containing it. Recall that we have another torus associated to Σ

$$T_\Sigma = T_Z/(T_Z \cap H_Z) = A_Z/(A_Z \cap H_Z)$$

such that $P(\Sigma) = X^*(T_\Sigma)$. We denoted $H = H_Z \cap (G \times \{1\})$. Let us put now $\tilde{H} = \mathrm{pr}_1(H_Z)$. Then,

$$T_Z \cap H_Z \subset (T \cap \tilde{H}) \times \mathrm{pr}_2(H_Z),$$

so we have natural projections

$$\mathrm{pr}_1 : T_\Sigma \to A/(A \cap \tilde{H}) =: A_{\tilde{H}} \text{ and } \mathrm{pr}_2 : T_\Sigma \to Z/\mathrm{pr}_2(H_Z) = A_\Sigma.$$

and inclusions $i_1 : X^*(A_{\tilde{H}}) \hookrightarrow X^*(T_\Sigma)$ and $i_2 : X^*(A_\Sigma) \hookrightarrow X^*(T_\Sigma)$. Moreover, the kernel of the projection $pr_2 : T_\Sigma \to A_\Sigma$ is the torus

$$T_\Sigma \cap (G \times \{1\}) = T/(T \cap H) = A/(A \cap H) = A_H.$$

Therefore, we obtain a natural projection $p : X^*(T_\Sigma) \to X^*(A_H)$. Note that since Σ is affine we also have a natural inclusion $P_+(\Sigma) \subset P_+(O_\Sigma) = X^*(T_\Sigma)_+$.

Definition 1.6.1. The GIT quotient $\mathbb{A}_\Sigma := \Sigma /\!\!/ G$ is an A_Σ-toric variety, which we call the *abelianization* of Σ. The natural projection $\alpha_\Sigma : \Sigma \to \mathbb{A}_\Sigma$ is called the *abelianization map* of Σ.

Remark 1.6.2. The abelianization $\mathbb{A}_\Sigma$ is simply the toric variety

$$\mathbb{A}_\Sigma = \operatorname{Spec}\left(\bigoplus_{\chi \in P_+(\mathbb{A}_\Sigma)} \mathbb{C}[e^\chi] \right),$$

where $P_+(\mathbb{A}_\Sigma)$ is the intersection

$$P_+(\mathbb{A}_\Sigma) = P_+(\Sigma) \cap i_2(X^*(A_\Sigma)).$$

Definition 1.6.3. A *very flat symmetric embedding* is a simple affine symmetric embedding Σ such that the abelianization map $\alpha_\Sigma : \Sigma \to \mathbb{A}_\Sigma$ is dominant, flat and with integral fibres.

Guay's classification

We assume for the rest of the section that G is simply-connected and fix $\theta \in \mathrm{Aut}_2(G)$. As in the previous section, we fix a maximal θ-split torus $A \subset G$. We are interested in classifying very flat symmetric embeddings Σ such that the semisimple part of O_Σ is equal to G/G^θ. These have been determined by Guay [Gua01, Proposition 7].

Theorem 1.6.4 (Guay)**.** *A simple affine symmetric embedding Σ with $O'_\Sigma = G/G^\theta$ is very flat if and only if there exists a homomorphism*

$$\psi : X^*(A_{G^\theta}) \longrightarrow X^*(Z_\Sigma),$$

such that, for any $(a, z) \in H_Z \cap A_Z \subset (\tilde{H} \cap A) \times pr_2(H_Z)$ and for any $\chi \in X^*(A_{G^\theta})$, we have

$$a^\chi = z^{-\psi(\chi)};$$

and such that the weight semigroup $P_+(\Sigma)$ is of the form

$$P_+(\Sigma) = \{(\chi, \psi(\chi) + \eta) : \chi \in X^*(A_{G^\theta})_+, \eta \in P_+(\mathbb{A}_\Sigma)\}.$$

Remark 1.6.5. When Σ is very flat any function $f \in \mathbb{C}[G/G^\theta]$ with weight χ can be extended to a function $f_+ \in \mathbb{C}[\Sigma]$ with weight $(\chi, \psi(\chi))$. Indeed, just define

$$f_+(g, z) = z^{\psi(\chi)} f(g).$$

It is clear that f_+ has the desired weight. Now, since $\tilde{H}$ is a symmetric subgroup for θ, we have that any element $h \in \tilde{H}$ is of the form $h = ah_0$ for $h \in G^\theta$ and $a \in A \cap \tilde{H}$. Thus, if $(h,s) \in H_Z$, we have, for any $(g,z) \in G_Z$,

$$f_+(hg, zs) = s^{\psi(\chi)} z^{\psi(\chi)} f(hg) = s^{\psi(\chi)} a^\chi f_+(g,z) = f_+(g,z).$$

This allows us to describe Σ more explicitly. For each fundamental dominant weight $\varpi_i \in X^*(A_{G^\theta})$, consider the G-submodule $\mathbb{C}[G/G^\theta]_i$ of $\mathbb{C}[G/G^\theta]$ formed by functions with weight ϖ_i, and take $f_i^1, \dots, f_i^{m_i}$ a basis of $\mathbb{C}[G/G^\theta]_i$ as a $\mathbb{C}$-vector space. On the other hand, let $\gamma_1, \dots, \gamma_s$ be generators of $P_+(\mathbb{A}_\Sigma) \subset X^*(Z)$. We can now define the map

$$\begin{aligned}(f_+, \alpha_+) : O_\Sigma &\longrightarrow \left(\bigoplus_{i=1}^{l} \mathbb{A}^{m_i}\right) \oplus \mathbb{A}^s \\ (g,z)H_Z &\longmapsto ((f_{i+}^1(g,z), \dots, f_{i+}^{m_i}(g,z)), (z^{\gamma_1}, \dots, z^{\gamma_s})).\end{aligned}$$

The symmetric embedding Σ is then identified with the closure of the image of this map, $\Sigma = \overline{(f_+, \alpha_+)(O_\Sigma)}$.

In order to prove Guay's classification theorem, one needs to consider the following preorder on $P_+(\Sigma)$

$$\chi_1 \geq \chi_2 \text{ if } \chi_1 - \chi_2 \in -P_+(\mathbb{A}_\Sigma).$$

Let $M(\Sigma)$ denote the set of minimal elements under this preorder, that is

$$M(\Sigma) = \{\chi \in P_+(\Sigma) : \chi' \leq \chi \implies \chi \leq \chi'\}.$$

The result follows from a series of lemmas, of which we omit the proofs and refer to [Gua01, Lemmas 6 and 7].

Lemma 1.6.6. *The following statements are equivalent:*

1. $\alpha_\Sigma : \Sigma \to \mathbb{A}_\Sigma$ *is flat.*
2. $\mathbb{C}[\Sigma] \cong \mathbb{C}[O_\Sigma]_{M'(\Sigma)} \otimes \mathbb{C}[\mathbb{A}_\Sigma]$, *for* $M'(\Sigma)$ *a set of representatives of the quotient*
$$M(\Sigma)/(P_+(\mathbb{A}_\Sigma) \cap (-P_+(\mathbb{A}_\Sigma))).$$

Remark 1.6.7. Note that since the embedding $O_\Sigma \hookrightarrow \Sigma$ is dominant, the image of the restriction of $p : X^*(T_\Sigma) \to X^*(A_{G^\theta})$ to $M'(\Sigma)$ is $X^*(A_{G^\theta})_+$.

Lemma 1.6.8. *The fibres of* $\alpha_\Sigma : \Sigma \to \mathbb{A}_\Sigma$ *are integral if and only if* $M(\Sigma)$ *is a subsemigroup of* $P_+(\mathbb{A}_\Sigma)$.

Proof of Theorem 1.6.4. We show that if Σ is very flat then we have the desired consequences, the converse being immediate from the previous lemmas. Indeed if Σ is very flat, from the projections $M(\Sigma) \to M'(\Sigma)$ and $p : M'(\Sigma) \to X^*(A_{G^\theta})$ we can construct a surjective group homomorphism

$$M(\Sigma) - M(\Sigma) \longrightarrow X^*(A_{G^\theta}).$$

Since it is an epimorphism, this map has a right inverse, which is of the form

$$\begin{aligned} X^*(A_{G^\theta}) &\longrightarrow M(\Sigma) - M(\Sigma) \\ \chi &\longmapsto (\chi, \psi(\chi)), \end{aligned}$$

where ψ is a group homomorphism $\psi : X^*(A_{G^\theta}) \to X^*(Z_\Sigma)$ such that if $\chi \in X^*(A_{G^\theta})_+$ then $(\chi, \psi(\chi)) \in M'(\Sigma)$. Now, since $M(\Sigma) - M(\Sigma)$ lies inside of the torus $T_\Sigma = A_Z/(H_Z \cap A_Z)$, for any $(a, z) \in H_Z \cap A_Z$ we must have

$$a^\chi z^{\psi(\chi)} = 1.$$

In conclusion, we get

$$M'(\Sigma) = \{(\chi, \psi(\chi)) : \chi \in X^*(A_{G^\theta})_+\}.$$

Now,

$$\mathbb{C}[O_\Sigma]_{(\chi,\psi(\chi))} = \mathbb{C}[G/G^\theta]_\chi \otimes \mathbb{C}[e^{\psi(\chi)}].$$

From this and the isomorphism $\mathbb{C}[\Sigma] \cong \mathbb{C}[O_\Sigma]_{M'(\Sigma)} \otimes \mathbb{C}[\mathbb{A}_\Sigma]$, the result follows. □

Amongst all simple affine symmetric embeddings, those having a fixed point will be of particular interest. By a fixed point of an Σ of a symmetric variety G/H we mean a point $0 \in \Sigma$ such that $g \cdot 0 = 0$ for any $g \in G$. Since by assumption Σ is simple, the set $\{0\} \subset \Sigma$ is its unique closed orbit and thus a fixed point, if it exists, is unique. From Guay's classification we can obtain necessary and sufficient conditions for the existence of a fixed point.

Proposition 1.6.9. *A very flat symmetric embedding Σ with $O'_\Sigma = G/G^\theta$ has a fixed point if and only if $P_+(\mathbb{A}_\Sigma)$ is pointed (meaning that if χ and $-\chi$ are in $P_+(\mathbb{A}_\Sigma)$, then $\chi = 0$) and the only element in $P_+(\Sigma)$ of the form $(\chi, \psi(\chi))$ for $\chi \in X^*(A_{G^\theta})_+$ is $(0, 0)$.*

Proof. A fixed point Σ corresponds to a maximal ideal $I \subset \mathbb{C}[\Sigma]$ which is fixed under the action of G_Z. If such an ideal exists, it must be of the form

$$I = \bigoplus_{(\chi,\psi(\chi)+\eta)\in P_+(\Sigma)\setminus\{(0,0)\}} \mathbb{C}[O_\Sigma]_{(\chi,\psi(\chi)+\eta)},$$

and this is only an ideal if for every $(\chi, \psi(\chi) + \eta) \in P_+(\Sigma) \setminus \{(0, 0)\}$, $(0, 0)$ is not a highest weight of $\mathbb{C}[O_\Sigma]_{(\chi,\psi(\chi)+\eta)}$. That is the case and $(0, 0)$ is a highest weight of $\mathbb{C}[O_\Sigma]_{(\chi,\psi(\chi)+\eta)}$ if and only if both η and $-\eta$ are contained in $P_+(\mathbb{A}_\Sigma)$. □

The enveloping embedding

By the universal property of categorical quotients, a morphism $\Sigma_1 \to \Sigma_2$ of simple affine symmetric embeddings induces a commutative square

$$\begin{array}{ccc} \Sigma_1 & \longrightarrow & \Sigma_2 \\ \downarrow & & \downarrow \\ \mathbb{A}_1 & \longrightarrow & \mathbb{A}_2. \end{array}$$

Definition 1.6.10. A morphism $\Sigma_1 \to \Sigma_2$ of simple affine symmetric embeddings is *excellent* if the induced square is Cartesian.

We denote by $\mathcal{VF}(G/G^\theta)$ the category with very flat symmetric embeddings with semisimple part G/G^θ as objects and excellent morphisms as arrows. We can also consider the subcategory $\mathcal{VF}_0(G/G^\theta)$ formed by very flat symmetric embeddings with a fixed point.

Suppose that θ does not have imaginary roots (that is $\Phi_{\mathfrak{g}}^\theta = \emptyset$, for $T \subset G$ a θ-stable maximal torus). For example, we can assume that θ is quasisplit. In that case, a versal object of $\mathcal{VF}(G/G^\theta)$ was constructed by Guay [Gua01]. We recall here his construction.

We begin by taking $A \subset T \subset G$ a maximal θ-split torus of G and putting $G_A = G \times A$ and

$$H_A = \left\{(nh, an^{-1}) : h \in G^\theta, a \in F^\theta, n \in F_\theta\right\},$$

where we recall that we denoted $F^\theta = G^\theta \cap A$ and $F_\theta = G_\theta \cap A$. The space $(G/G^\theta)_+ := G_A/H_A$ is a symmetric variety associated to the involution $\vartheta : (g, a) \mapsto (\theta(g), a^{-1})$ with semisimple part isomorphic to G/G^θ. Indeed, we have

$$H = H_A \cap (G \times \{1\}) = G^\theta, \quad \text{and} \quad \tilde{H} = \mathrm{pr}_1(H_A) = G_\theta.$$

A maximal ϑ-split torus of G_A is given by $A \times A$.

Definition 1.6.11. We define the (Guay) *enveloping embedding* of G/G^θ as the symmetric embedding $\mathrm{Env}(G/G^\theta)$ with $O_{\mathrm{Env}(G/G^\theta)} = (G/G^\theta)_+$ and determined by the weight semigroup

$$P_+(\mathrm{Env}(G/G^\theta)) = \{(\chi, w_0\chi + \eta) : \chi \in X^*(A_{G^\theta})_+, \eta \in -\mathbb{Z}_+\langle\Delta_\theta\rangle\} \cup \{(0,0)\},$$

for $\Delta_\theta = \{\bar{\alpha}_1, \dots, \bar{\alpha}_l\}$ the simple restricted roots associated to θ.

Note that $Z_{\mathrm{Env}(G/G^\theta)} = A/F^\theta = A_{G^\theta}$ and $A_{\mathrm{Env}(G/G^\theta)} = A/F_\theta = A_{G_\theta}$. Now, we can define

$$\begin{aligned} \psi : X^*(A_{G^\theta}) &\longrightarrow X^*(A_{G^\theta}) \\ \chi &\longmapsto w_0\chi. \end{aligned}$$

An element of $H_A \cap (A \times A)$ is of the form (na_1, a_2n^{-1}) for $a_1, a_2 \in F^\theta$ and $n \in F_\theta$. Therefore,

$$(na_1)^{-w_0\chi} = n^{-w_0\chi} = (n^{-1})^{w_0\chi} = (a_2n^{-1})^\chi.$$

We conclude that $\mathrm{Env}(G/G^\theta)$ is a very flat symmetric embedding. Moreover, one checks easily that it has a fixed point $0 \in \mathrm{Env}(G/G^\theta)$. Its complement $\mathrm{Env}^0(G/G^\theta) = \mathrm{Env}(G/G^\theta) \setminus \{0\}$ is a smooth dense subvariety and it follows from [Gua01, Theorem 4] that the GIT quotient $\mathrm{Env}^0(G/G^\theta) /\!\!/ A_{G^\theta}$ can be shown to be isomorphic to the wonderful compactification $\overline{G/G_\theta}$.

The main result of Guay's paper is the following [Gua01, Theorem 3].

Theorem 1.6.12 (Guay). *When $\Phi_{\mathfrak{g}}^{\theta} = \varnothing$, the envelopping embedding* $\mathrm{Env}(G/G^{\theta})$ *is a versal object of the category* $\mathcal{VF}(G/G^{\theta})$ *and a universal object of* $\mathcal{VF}_0(G/G^{\theta})$*. That is, for any very flat symmetric embedding* Σ *with semisimple part* G/G^{θ} *there exists an excellent morphism* $\Sigma \to \mathrm{Env}(G/G^{\theta})$*, which is unique if* Σ *has a fixed point.*

Proof. If Σ is a very flat symmetric embedding, then there exists a homomorphism $\psi : X^*(A_{G^{\theta}}) \to X^*(Z_{\Sigma})$ as in the statement of Theorem 1.6.4. This homomorphism can be extended to a homomorphism $\tilde{\psi} : X^*(T_{\mathrm{Env}(G/G^{\theta})}) \to X^*(T_{\Sigma})$. Indeed, the inclusion $G_A^{\vartheta} \subset H_A$ induces a surjection from $(A \times A)/((A \times A) \cap G_A^{\vartheta}) = A_{G^{\theta}} \times A_{G^{\theta}}$ onto $(A \times A)/((A \times A) \cap H_A) = T_{\mathrm{Env}(G/G^{\theta})}$, and thus a homomorphism $X^*(T_{\mathrm{Env}(G/G^{\theta})}) \to X^*(A_{G^{\theta}}) \oplus X^*(A_{G^{\theta}})$. The map $\tilde{\psi}$ is then defined as

$$X^*(T_{\mathrm{Env}(G/G^{\theta})}) \longrightarrow X^*(A_{G^{\theta}}) \oplus X^*(A_{G^{\theta}}) \xrightarrow{(\mathrm{id}, \psi \circ w_0)} X^*(A_{G^{\theta}}) \oplus X^*(Z_{\Sigma}) \xrightarrow{(i_1, i_2)} X^*(T_{\Sigma}).$$

We claim that $\tilde{\psi}(P_+(\mathrm{Env}(G/G^{\theta}))) \subset P_+(\Sigma)$. This induces a homomorphism $\mathbb{C}[\mathrm{Env}(G/G^{\theta})] \hookrightarrow \mathbb{C}[\Sigma]$ and dually a morphism $\Sigma \to \mathrm{Env}(G/G^{\theta})$. To prove our claim, it suffices to show that for any simple restricted root $\bar{\alpha}_i \in \Delta_{\theta}$, the character $-\psi(w_0\bar{\alpha}_i)$ belongs to the weight semigroup $P_+(\mathbb{A}_{\Sigma})$. This follows from the following lemma; we refer to [Gua01, Lemma 11] for a proof.

Lemma 1.6.13. *Let ϖ_i be a fundamental dominant weight associated to Δ_{θ} of the form $\varpi_i = \omega_i + \omega_{\sigma(i)}$, where ω_i is a fundamental dominant weight of* T *and σ is defined as in Section 1.3. Then the* G*-module $V_{\varpi_i} \otimes V_{\varpi_i}$ contains the irreducible representation* $V_{2\varpi_i - \bar{\alpha}_i}$.

Since we assumed that $\Phi_{\mathfrak{g}}^{\theta} = \varnothing$, the lemma above is satisfied for every simple restricted root. Now, from the lemma it follows that both $(2\varpi_i, \psi(2\varpi_i))$ and $(2\varpi_i - \bar{\alpha}_i, \psi(2\varpi_i))$ belong to the weight semigroup $P_+(\Sigma)$. But

$$(2\varpi_i - \bar{\alpha}_i, \psi(2\varpi_i)) = (2\varpi_i - \bar{\alpha}_i, \psi(2\varpi_i - \bar{\alpha}_i) + \psi(\bar{\alpha}_i)),$$

so $\psi(\bar{\alpha}_i) \in P_+(\mathbb{A}_{\Sigma})$ and thus $-\psi(w_0\bar{\alpha}_i) \in P_+(\mathbb{A}_{\Sigma})$. That the resulting morphism is excellent follows easily from [Vin95, Proposition 7]. □

Invariant theory of symmetric varieties

Let G be a reductive group and $\theta \in \mathrm{Aut}_2(G)$ an involution. The action of G on G/G^{θ} by left multiplication restricts to an action of G^{θ}. This action was widely studied by Richardson, and we state here his main result [Ric82b, Corollary 11.5].

Theorem 1.6.14 (Richardson). *Let $A \subset G$ be a maximal θ-split torus and W_{θ} the little Weyl group. The restriction homomorphism $\mathbb{C}[G/G^{\theta}] \to \mathbb{C}[A_{G^{\theta}}]$ induces an isomorphism*

$$\mathbb{C}[G/G^{\theta}]^{G^{\theta}} \xrightarrow{\sim} \mathbb{C}[A_{G^{\theta}}]^{W_{\theta}}.$$

We denote

$$\mathfrak{c}_{G/G^{\theta}} = (G/G^{\theta}) /\!\!/ G^{\theta} \cong A_{G^{\theta}}/W_{\theta}.$$

Richardson's result can be extended easily to a very flat symmetric embedding. Indeed, suppose that G is semisimple simply-connected and let Σ be a very flat symmetric embedding with $O'_\Sigma = G/G^\theta$. The left action of G^θ extends to an action on Σ. Let us denote $\mathbf{c}_\Sigma = \Sigma /\!\!/ G^\theta$.

Proposition 1.6.15. *If Σ is very flat, there is an isomorphism*

$$\mathbf{c}_\Sigma = \mathbf{c}_{G/G^\theta} \times \mathbb{A}_\Sigma.$$

Proof. Since every element of $P_+(\Sigma)$ is of the form $(\chi, \psi(\chi) + \eta)$ for $\chi \in X^*(A_{G^\theta})_+$ and $\eta \in P_+(\mathbb{A}_\Sigma)$, we have

$$\mathbb{C}[\Sigma]^{G^\theta} = \bigoplus_{(\chi,\psi(\chi)+\eta)\in P_+(\Sigma)} \mathbb{C}[O_\Sigma]^{G^\theta}_{(\chi,\psi(\chi)+\eta)} = \mathbb{C}[G/G^\theta]^{G^\theta} \otimes \mathbb{C}[\mathbb{A}_\Sigma],$$

that is, $\mathbb{C}[\mathbf{c}_\Sigma] = \mathbb{C}[\mathbf{c}_{G/G^\theta}] \otimes \mathbb{C}[\mathbb{A}_\Sigma]$. □

It follows from the discussion in [Ric82b, Sections 13 and 14] that when G is semisimple simply-connected, the ring $\mathbb{C}[A_{G^\theta}]^{W_\theta}$ is a polynomial algebra and thus $\mathbb{A}_{G^\theta}/W_\theta$ is an affine space. Moreover, since $\mathbb{C}[A_{G^\theta}] = \mathbb{C}[e^{X^*(A_{G^\theta})}]$ and the lattice $X^*(A_{G^\theta})$ is generated by the fundamental weights $\varpi_1, \dots, \varpi_l$, we get an isomorphism

$$\mathbb{C}[\mathbf{c}_{G/G^\theta}] \cong \mathbb{C}[b_1, \dots, b_l],$$

where each b_i is a function in $\mathbb{C}[G/G^\theta]$ with weight ϖ_i. In general, for G semisimple not simply-connected, Richardson [Ric82b, Section 15] characterizes for which involutions the ring $\mathbb{C}[A_{G^\theta}]^{W_\theta}$ is a polynomial algebra

1.7 FORMAL LOOP PARAMETRIZATION

Formal arc and loop spaces

Let $\mathcal{O} = \mathbb{C}[[z]]$ denote the ring of formal power series in a formal variable z and $F = \mathrm{qf}(\mathcal{O}) = \mathbb{C}((z))$ its quotient field, the field of formal Laurent series in z. Given any $\mathbb{C}$-scheme Σ, a *formal loop* on Σ is a morphism of $\mathbb{C}$-schemes $\operatorname{Spec} F \to \Sigma$, while a *formal arc* on Σ is a morphism of $\mathbb{C}$-schemes $\operatorname{Spec} \mathcal{O} \to \Sigma$. We denote by $\Sigma(F)$ and $\Sigma(\mathcal{O})$ the sets of formal loops and formal arcs on Σ, respectively. Both these sets can be regarded as the spaces of $\mathbb{C}$-points of the functors Σ_F and $\Sigma_\mathcal{O}$ sending any $\mathbb{C}$-algebra R to $\Sigma(R \otimes_\mathbb{C} F)$ and $\Sigma(R \otimes_\mathbb{C} \mathcal{O})$, respectively. These functors can be endowed with the structure of an *ind-scheme*.

If G is an algebraic group, the space G_F is called the *formal loop group* and $G_\mathcal{O}$ is the *formal arc group*. The homogeneous space $\mathrm{Gr}_G = G_F/G_\mathcal{O}$ is known as the *affine Grassmannian* of G.

The Cartan decomposition

Let G be a reductive algebraic group over $\mathbb{C}$ and $T \subset G$ a maximal torus. Any cocharacter $\lambda \in X_*(T)$ defines a formal loop $z^\lambda \in G(F)$ by taking the image of the formal variable $z \in F$ under the induced homomorphism

$$\lambda(F) : F^{\times} \longrightarrow T(F) \subset G(F).$$

We now have the following result, due to Iwahori and Matsumoto [IM65], but usually called the Cartan decomposition of $G(F)$.

Proposition 1.7.1. *If we fix a Borel subgroup* $B \subset G$ *containing* T*, we can decompose*

$$G(F) = \bigsqcup_{\lambda \in X_*(T)_+} G(\mathcal{O}) z^{\lambda} G(\mathcal{O}).$$

Proof. This is a consequence of the Bruhat decomposition of Tits systems. Indeed, one can take the group $\mathfrak{G} = G(F) \rtimes \mathbb{C}^*$, the subgroups $\mathfrak{B}$, defined as the inverse image of B under the evaluation map $G(\mathcal{O}) \rtimes \mathbb{C}^* \to G$, and $\mathfrak{N} = z^{X_*(T)} \rtimes (N_G(T) \times \mathbb{C}^*)$, and the subset $\mathfrak{S}$ of natural generators of the affine Weyl group $\tilde{W} = X_*(T) \rtimes W$. It is easy to check that $(\mathfrak{G}, \mathfrak{B}, \mathfrak{N}, \mathfrak{S})$ satisfies the required axioms and thus forms a Tits system, and its Weyl group is precisely $\tilde{W}$. Moreover, one can take as a set of representatives of $\tilde{W}$ in $\mathfrak{N}$ the dominant cocharacter semigroup $X_*(T)_+$. Therefore, the Bruhat decomposition of this Tits system gives

$$G(F) \rtimes \mathbb{C}^* = \mathfrak{G} = \bigsqcup_{\lambda \in X_*(T)_+} \mathfrak{B} z^{\lambda} \mathfrak{B},$$

from where the result follows. □

By using the properties of Tits systems (or as a consequence of our later Proposition 1.7.6), one can also prove the following.

Proposition 1.7.2. *The closure of any orbit* $G(\mathcal{O}) z^{\lambda} G(\mathcal{O})$ *inside* $G(F)$ *is equal to*

$$\overline{G(\mathcal{O}) z^{\lambda} G(\mathcal{O})} = \bigsqcup_{\mu \in X_*(T), \mu \leq \lambda} G(\mathcal{O}) z^{\mu} G(\mathcal{O}).$$

Loop parametrization of a symmetric variety

The results stated above for reductive groups can be easily generalized to symmetric varieties. We thus begin by taking a reductive algebraic group G over $\mathbb{C}$, an involution $\theta \in \mathrm{Aut}_2(G)$, a maximal θ-split torus $A \subset G$, a maximal torus $T \subset G$ containing A, and a Borel subgroup $B \subset G$ contained in a minimal θ-split parabolic subgroup and containing T.

Proposition 1.7.3. *Given any symmetric subgroup* $H \subset G$ *associated to* θ*, we can decompose*

$$(G/H)(F) = \bigsqcup_{\lambda \in X_*(A_H)_-} G(\mathcal{O}) z^{\lambda}.$$

Remark 1.7.4. Note that z^{λ} is a well defined element of $G/H(F)$ since there is a natural inclusion $A_H = T/(T \cap H) \hookrightarrow G/H$.

Proof. We begin by reducing to the case $H = G_\theta$. Thus, suppose that there is an element of $z^{X_*(A_{G_\theta})_-}$ in the orbit of every element of $(G/G_\theta)(F)$. Since, by definition of a symmetric subgroup, we have $H \subset G_\theta$, there is a natural projection $G/H \to G/G_\theta$ and thus $(G/H)(F) \to (G/G_\theta)(F)$. Moreover, since the inverse image of A_{G_θ} under the projection $G/H \to G/G_\theta$ is A_H, the inverse image of $z^{X_*(A_{G_\theta})_-}$ is $z^{X_*(A_H)_-}$. Since the projection is $G(\mathcal{O})$-equivariant, we conclude that there is an element of $z^{X_*(A_H)_-}$ in the orbit of every element of $(G/H)(F)$. Uniqueness follows from the fact that anti-dominant weights of A_H are anti-dominant for T and from the Cartan decomposition of $G(F)$.

It remains to show that for every $\phi \in (G/G_\theta)(F)$ there exists some $\lambda \in X_*(A_{G_\theta})_-$ such that z^λ is in the $G(\mathcal{O})$-orbit of ϕ. Consider then the wonderful compactification $\overline{G/G_\theta}$ of G/G_θ. Since it is projective, by the valuative criterion of properness, every formal loop $\phi \in (G/G_\theta)(F)$ extends to a formal arc $\bar{\phi} \in \overline{G/G_\theta}(\mathcal{O})$. By the local structure theorem, there exists some unipotent group P^u, and some formal arcs $g \in G(\mathcal{O})$, $u \in P^u(\mathcal{O})$ and $\bar{a} \in \mathbb{A}(\mathcal{O})$ such that

$$\bar{\phi} = gu\bar{a}.$$

Here, we recall that $\mathbb{A}$ denoted the A-toric variety

$$\mathbb{A} = \operatorname{Spec}\left(\bigoplus_{\bar{\alpha}\in\Delta_\theta} \mathbb{C}[e^{-2\bar{\alpha}}]\right),$$

which can be identified with the l-dimensional affine space $\mathbb{A}^l$ in such a way that the embedding $A \hookrightarrow \mathbb{A}$ is given explicitly as $a \mapsto (a^{-2\bar{\alpha}_1}, \dots, a^{-2\bar{\alpha}_l})$. Therefore, a cocharacter $\lambda : \mathbb{C}^* \to A$ extends to a morphism $\mathbb{A}^1 \to \mathbb{A}$ if and only if $\langle \lambda, 2\bar{\alpha}_i\rangle \leq 0$ for every $i = 1, \dots, l$, that is, if $\lambda \in \mathcal{P}^\vee_{\theta,-} = X_*(A_{G_\theta})_-$. Therefore, $\bar{a} = az^\lambda$ for some $\lambda \in X_*(A_{G_\theta})_-$. □

Remark 1.7.5. Let $H \subset G$ be a symmetric subgroup associated to θ and $f \in \mathbb{C}[G/H]_\chi$ a function with weight $\chi \in X^*(A_H)_+$. Recall that for any dominant weight χ, if V_χ is the irreducible representation with highest weight χ, we have an isomorphism $V_\chi^* = V_{-w_0\chi}$, for w_0 the longest element of the Weyl group, so $f(gt) = t^{-w_0\chi} f(g)$ for any $g \in G$ and $t \in T$. Therefore, if $\phi = gz^\lambda \in G(\mathcal{O})z^\lambda$, we have

$$f(\phi) = (z^\lambda)^{-w_0\chi} f(g) = z^{-\langle\lambda, w_0\chi\rangle} f(g).$$

Thus, $f(\phi)$ is a Laurent series with highest pole order less or equal than $\langle\lambda, w_0\chi\rangle$.

Let us describe now the closures of the orbits $G(\mathcal{O})z^\lambda$. Recall that we can define an order in $X(A_H)_-$ by putting $\lambda \leq \mu$ if and only if $\lambda - \mu \in -\mathbb{N}\langle\Delta_\theta^\vee\rangle$.

Proposition 1.7.6. *For any* $\lambda \in X_*(A_H)_-$, *the closure of* $G(\mathcal{O})z^\lambda$ *is equal to*

$$\overline{G(\mathcal{O})z^\lambda} = \bigsqcup_{\mu\in X_*(A_H)_-, \mu\leq\lambda} G(\mathcal{O})z^\mu.$$

Proof. Let $f \in \mathbb{C}[G/H]_\chi$ be a function with weight $\chi \in X^*(A_H)_+$. If $\phi \in G(\mathcal{O})z^\mu$ is such that ϕ belongs to the closure of $G(\mathcal{O})z^\lambda$ then the highest pole order of $f(\phi)$ must be less or equal than $\langle \lambda, w_0\chi\rangle$, so

$$\langle \mu, w_0\chi\rangle \le \langle \lambda, w_0\chi\rangle \implies \langle \lambda-\mu, w_0\chi\rangle \ge 0.$$

Now, since $w_0X^*(A_H)_+ = X^*(A_H)_-$, we conclude that $\lambda-\mu \in -\mathbb{N}\langle \Delta_\theta^\vee\rangle$.

Reciprocally, we want to show that for every $\mu \le \lambda$, $G(\mathcal{O})z^\mu \in \overline{G(\mathcal{O})z^\lambda}$. It suffices to find an element $\phi \in G(\mathcal{O})z^\mu$ with $\phi \in \overline{G(\mathcal{O})z^\lambda}$. The argument is analogous to the one given for the affine Grassmannian (see [Zhu17]). We do it for $G = SL_2$ and the general result follows by considering the canonical homomorphism $SL_2 \to G$ associated to the root $\bar\alpha$ with $\lambda - \mu = \bar\alpha^\vee$. Given any $m \in \mathbb{N}$ we can consider the family

$$\begin{pmatrix} z^m & 0 \\ z^{-m} + t^{-1}z^{-m+1} & z^{-m}\end{pmatrix} \in SL_2(\mathcal{O})z^{-m},$$

for $t \in \mathbb{C}^*$. One can easily check that

$$\begin{pmatrix} t^{-1} & 0 \\ 0 & t\end{pmatrix}\begin{pmatrix} 1 & -tz^{2m-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} z^m & 0 \\ z^{-m} + t^{-1}z^{-m+1} & z^{-m}\end{pmatrix} = \begin{pmatrix} -z^{m-1} & -z^{m-1} \\ tz^{-m} + z^{-m+1} & tz^{-m}\end{pmatrix},$$

which lies in the same orbit $SL_2(\mathcal{O})z^{-m}$ as the original matrix. Now, the limit as $t \to 0$ clearly has to lie in the closure $\overline{SL_2(\mathcal{O})z^{-m}}$, but this limit is

$$\begin{pmatrix} -z^{m-1} & -z^{m-1} \\ z^{-m+1} & 0\end{pmatrix} = \begin{pmatrix} -z^{2m} & -z^{2m} \\ 1 & 0\end{pmatrix} z^{-m+1} \in SL_2(\mathcal{O})z^{-(m-1)}$$

We conclude that $SL_2(\mathcal{O})z^{-(m-1)} \subset \overline{SL_2(\mathcal{O})z^{-m}}$ □

Loop parametrization of the enveloping embedding

Suppose now that G is semisimple simply-connected. We describe now the formal loops on the enveloping embedding $\mathrm{Env}(G/G^\theta)$.

We begin by noticing, as in the proof of Proposition 1.7.3 that if $\mathbb{A}$ is an A-toric variety with weight semigroup $P_+(\mathbb{A}) \subset A$, then an element $a \in A(F)$ extends to an element $\bar a \in \mathbb{A}(\mathcal{O})$ if and only if $a \in A(\mathcal{O})z^\lambda$, for λ in the dual semigroup

$$P_+(\mathbb{A})^\vee = \{\lambda \in X^*(A) : \langle \lambda, \chi\rangle \in \mathbb{N}, \forall \chi \in P_+(\mathbb{A})\}.$$

In other words,

$$A(F) \cap \mathbb{A}(\mathcal{O}) = \bigsqcup_{\lambda \in P_+(\mathbb{A})^\vee} A(\mathcal{O})z^\lambda.$$

In particular, we get

$$A_{G_\theta}(F) \cap \mathbb{A}_{\mathrm{Env}(G/G^\theta)}(\mathcal{O}) = \bigsqcup_{\lambda \in X^*(A_{G_\theta})_-} A_{G_\theta}(\mathcal{O})z^\lambda,$$

since $X^*(A_{G_\theta})_- = (-\mathbb{N}\langle\Delta_\theta\rangle)^\vee$.

Consider an anti-dominant cocharacter $\lambda \in X_*(A_{G_\theta})_-$ and define $\mathrm{Env}^\lambda(G/G^\theta)$ as the fibered product

$$\begin{array}{ccc} \mathrm{Env}^{\lambda}(G/G^{\theta}) & \longrightarrow & A_{G_\theta} \times \mathrm{Env}(G/G^{\theta}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,\mathcal{O} & \xrightarrow{z^{-w_0\lambda}} & \mathbb{A}_{\mathrm{Env}(G/G^{\theta})}, \end{array}$$

where the vertical map on the right is the multiplication of the abelianization map $\alpha_{\mathrm{Env}(G/G^{\theta})}$ with the natural embedding $A_{G_\theta} \hookrightarrow \mathbb{A}_{\mathrm{Env}(G/G^{\theta})}$. Replacing $\mathrm{Env}(G/G^{\theta})$ by the open subvariety $\mathrm{Env}^0(G/G^{\theta})$ we define an open subvariety $\mathrm{Env}^{\lambda,0}(G/G^{\theta})$. The above stratification induces

$$\mathrm{Env}(G/G^{\theta})(\mathcal{O}) \cap (G/G^{\theta})_+(F) = \bigsqcup_{\lambda \in X^*(A_{G_\theta})_-} \mathrm{Env}^{\lambda}(G/G^{\theta})(\mathcal{O}).$$

We also note that

$$\mathrm{Env}^{\lambda,0}(G/G^{\theta})(\mathcal{O}) = \mathrm{Env}^{\lambda}(G/G^{\theta})(\mathcal{O}) \cap \mathrm{Env}^{0}(G/G^{\theta})(\mathcal{O})$$

Proposition 1.7.7. *For any $\phi \in (G/G^{\theta})_+(F)$, we have $\phi \in \mathrm{Env}^{\lambda}(G/G^{\theta})(\mathcal{O})$ if and only if the image of ϕ in $(G/G_\theta)(F)$ belongs to $\overline{G(\mathcal{O})z^{\lambda}}$. Moreover, $\phi \in \mathrm{Env}^{\lambda,0}(G/G^{\theta})(\mathcal{O})$ if and only if the image of ϕ in $(G/G_\theta)(F)$ belongs to $G(\mathcal{O})z^{\lambda}$.*

Proof. It suffices to show it for $\phi = (z^{\mu}, z^{\eta})$ with $\mu, \eta \in X_*(A_{G_\theta})_-$. First, note that by construction we must have $z^{\eta} = z^{-w_0\lambda}$, so we get $\eta = -w_0\lambda$. On the other hand, $\phi \in \mathrm{Env}(G/G^{\theta})(\mathcal{O})$ if and only if $f_+(\phi) \in \mathcal{O}$ for any $f \in \mathbb{C}[G/G^{\theta}]$. Thus, for any $\chi \in X^*(A_{G^{\theta}})_+$ and for any $f \in \mathbb{C}[G/G^{\theta}]_{\chi}$ we have

$$\mathcal{O} \ni f_+(z^{\mu}, z^{\eta}) = z^{\langle w_0\chi, \eta\rangle} z^{\langle \chi, \mu\rangle}.$$

Therefore,

$$0 \leq \langle w_0\chi, \eta\rangle + \langle \chi, \mu\rangle = \langle w_0\chi, -w_0\lambda\rangle + \langle \chi, \mu\rangle = \langle \chi, \mu - \lambda\rangle.$$

We have this for any $\chi \in X^*(A_{G^{\theta}})_+$, so $\mu - \lambda \in \mathbb{N}(\Delta_{\theta}^{\vee})$ and $\lambda \geq \mu$. Finally, $\phi \in \mathrm{Env}^0(G/G^{\theta})(\mathcal{O})$ if and only if $f_+(\phi) \in \mathcal{O}$ does not vanish, thus, if and only if $0 \geq \langle \chi, \mu - \lambda\rangle$, so $\lambda = \mu$. □

1.8 THE THEORY OF REDUCTIVE MONOIDS

Reductive monoids

As a particular example of the theory of symmetric embeddings developed in the previous sections, we can obtain the theory of reductive monoids. We recall the main results here, since we use them in the following.

Definition 1.8.1. An *algebraic semigroup* over $\mathbb{C}$ is a semigroup object in the category of $\mathbb{C}$-schemes, that is, a $\mathbb{C}$-scheme M endowed with an associative multiplication $\cdot : M \times M \to M$. When M has a multiplicative identity $1 \in M(\mathbb{C})$, it is an *algebraic monoid*. Invertible elements form an algebraic group $M^{\times} \hookrightarrow M$. When $M^{\times}$ is reductive, we say that M is a *reductive monoid*.

Let G be a reductive group over $\mathbb{C}$. It is a well-known result of Rittatore (see [Tim11, Theorem 27.5] for a proof) that monoids with $M^\times = G$ are the same as simple affine $(G\times G)$-equivariant embeddings of G under the multiplication action

$$(g_1, g_2) \cdot g = g_1 g g_2.$$

Recall from the diagonal example, Example 1.2.5, that G can be identified with the symmetric variety $(G \times G)/\Delta(G)$. Guay's theory of symmetric embeddings [Gua01] is modelled after the theory of reductive monoids as developed originally by Vinberg [Vin95].

Abelianization

Let us begin by describing the abelianization of a monoid, as a particular case of the abelianization of a symmetric embedding. We start by considering a reductive monoid M. Its unit group $M^\times$ is a reductive group and thus it can be decomposed as a product $M^\times = GZ$, where $G = (M^\times)'$ is the derived group of $M^\times$ and $Z = Z^0_{M^\times}$ is its connected centre, which is a torus. We put $A_M = Z/(Z \cap G)$.

If we let $T \subset G$ be a maximal torus of G, $TZ \subset M^\times$ is a maximal torus of $M^\times$. Now, $P(M) = X^*(TZ)$ and $P_+(M) \subset X^*(TZ)_+$. Now, $Z = (TZ)/Z$ and $A_M = (TZ)/((TZ) \cap T)$, so we have a natural inclusion $X^*(A_M) \hookrightarrow X^*(TZ)$.

The abelianization of M is the GIT quotient $\mathbb{A}_M := M /\!\!/ (G' \times G')$, and the natural projection $\alpha_M : M \to \mathbb{A}_M$ is the abelianization map. This quotient $\mathbb{A}_M$ can also be understood as a A_M-toric variety, where $Z = Z^0_G$ is the connected centre of G. Indeed, $\mathbb{A}_M$ is the toric variety with weight semigroup

$$P_+(\mathbb{A}_M) = P_+(M) \cap X^*(A_M).$$

Definition 1.8.2. A monoid M is *very flat* if the abelianization map $\alpha_M : M \to \mathbb{A}_M$ is dominant, flat and with integral fibres.

Vinberg's classification

Guay's classification of symmetric embeddings is a generalization of Vinberg's classification of reductive monoids [Vin95, Theorem 4]. We assume now that G is a semisimple simply-connected group.

Theorem 1.8.3 (Vinberg)**.** *A reductive monoid* M *with* $(M^\times)' = G$ *is very flat if and only if there exists a homomorphism*

$$\psi : X^*(T) \longrightarrow X^*(Z),$$

such that, for any $(t, z) \in T \times Z$ *and for any* $\chi \in X^*(T)$, *we have*

$$t^\chi = z^{-\psi(\chi)};$$

and such that the weight semigroup $P_+(M)$ *is of the form*

$$P_+(M) = \{(\chi, \psi(\chi) + \eta) : \chi \in X^*(T), \eta \in P_+(\mathbb{A}_M)\}.$$

A fixed point of a monoid is just a zero element in it, that is, an element $0 \in M$ such that $x0 = 0x = 0$ for every $x \in M$. If a monoid has a zero, then it is unique.

The enveloping monoid

A morphism of monoids $M_1 \to M_2$ is excellent if it is excellent as a morphism of symmetric embeddings, that is, if it induces a Cartesian diagram

$$\begin{array}{ccc} M_1 & \longrightarrow & M_2 \\ \downarrow & & \downarrow \\ \mathbb{A}_1 & \longrightarrow & \mathbb{A}_2. \end{array}$$

The category $\mathcal{VF}(G)$ is the category of very flat monoids M with $(M^\times)' = G$. We can also consider the category $\mathcal{VF}_0(G)$ of very flat monoids with a zero element. The Guay construction in this case yields the (Vinberg) enveloping monoid Env(G), which is determined by the unit group $G_+ = (G\times T)/Z_G$ and the weight semigroup

$$P_+(\mathrm{Env}(G)) = \left\{(\chi, w_0\chi + \eta) : \chi \in X^*(T)_+, \eta \in -\mathbb{Z}_+\langle \Delta_\mathfrak{g}\rangle\right\} \cup \{(0,0)\}.$$

This is clearly a very flat monoid with zero. The complement $\mathrm{Env}^0(G) = \mathrm{Env}(G) \setminus \{0\}$ is a smooth open subvariety of Env(G) and the GIT quotient $\mathrm{Env}^0(G) /\!\!/ T$ is isomorphic to the wonderful compactification $\overline{G^{ad}}$ of G^{ad}. The good properties of the Vinberg monoid are given in the following [Vin95, Theorem 5].

Theorem 1.8.4 (Vinberg). *The enveloping monoid* Env(G) *is a versal object of the category* $\mathcal{VF}(G)$ *and a universal object of* $\mathcal{VF}_0(G)$.

The invariant theory of a monoid

Richardson's theorem 1.6.14 is a generalization of the (multiplicative version of) the Chevalley restriction theorem. This theorem says that the restriction homomorphism $\mathbb{C}[G] \to \mathbb{C}[T]$ induces an isomorphism $\mathbb{C}[G]^G \xrightarrow{\sim} \mathbb{C}[T]^{W_T}$, where G acts on itself through the adjoint action and W_T is the Weyl group of T. We denote $\mathbf{c}_G = G /\!\!/ G \cong T/W$.

If G is semisimple simply-connected, since $\mathbb{C}[T] = \mathbb{C}[e^{X^*(T)}]$ and the lattice $X^*(T)$ is generated by the fundamental weights of the root system $\Phi_\mathfrak{g}$ of T, there is an isomorphism

$$\mathbb{C}[\mathbf{c}_G] \cong \mathbb{C}[b_1, \ldots, b_r],$$

where r is the rank of G and each $b_i = \mathrm{tr}(\rho_i)$ is the trace of the fundamental representation $\rho_i : G \to GL(V_{\omega_i})$ associated to the i-th fundamental weight ω_i.

The extension of the Chevalley isomorphism to a very flat monoid is a consequence of the results of Vinberg [Vin95]. Let us denote $\mathbf{c}_M = M /\!\!/ G$ under the adjoint action of G on M.

Proposition 1.8.5 (Vinberg). *If* M *is a very flat monoid, there is an isomorphism*

$$\mathbf{c}_M = \mathbf{c}_G \times \mathbb{A}_M.$$

Formal loop parametrization

The Cartan decomposition of the formal loop group $G(F)$ can be extended to an stratification of the enveloping monoid $\mathrm{Env}(G)$. First of all, note that

$$T^{\mathrm{ad}}(F) \cap \mathbb{A}_{\mathrm{Env}(G)}(\mathcal{O}) = \bigsqcup_{\lambda \in X^*(T)_+} T(\mathcal{O}) z^{-w_0\lambda},$$

since $X^*(T)_+ = (\mathbb{N}\langle\Delta_{\mathfrak{g}}\rangle)^\vee$. For any dominant cocharacter $\lambda \in X_*(T)_+$ we can define $\mathrm{Env}^\lambda(G)$ and $\mathrm{Env}^{\lambda,0}(G)$ as fibered products in the same way as we did for symmetric varieties and obtain

$$\mathrm{Env}(G)(\mathcal{O}) \cap G_+(F) = \bigsqcup_{\lambda \in X^*(T)} \mathrm{Env}^\lambda(G)(\mathcal{O}).$$

Proposition 1.7.7 in this case yields the following result of J. Chi [Chi22, Lemma 2.5.1].

Proposition 1.8.6 (J. Chi)**.** *For any $\phi \in G_+(F)$, we have $\phi \in \mathrm{Env}^\lambda(G)(\mathcal{O})$ if and only if the image of ϕ in $G^{\mathrm{ad}}(F)$ belongs to $\overline{G(\mathcal{O})z^\lambda G(\mathcal{O})}$. Moreover, $\phi \in \mathrm{Env}^{\lambda,0}(G)(\mathcal{O})$ if and only if the image of ϕ in $G^{\mathrm{ad}}(F)$ belongs to $G(\mathcal{O})z^\lambda G(\mathcal{O})$.*

2 MULTIPLICATIVE HIGGS BUNDLES AND INVOLUTIONS

2.1 MULTIPLICATIVE HIGGS BUNDLES

The generalized Hitchin map

Our study concerning Higgs pairs and Hitchin maps associated to symmetric varieties fits into a wider picture envisioned by Morrissey and Ngô [MN, Ng23]. We explain here their point of view.

Let Σ be an affine variety over $\mathbb{C}$ endowed with two commuting actions under two reductive algebraic groups H and Z. In other words, Σ is acted on by the product $H \times Z$. We let $\mathbf{c}_\Sigma$ denote the GIT quotient

$$\mathbf{c}_\Sigma = \Sigma /\!\!/ H = \operatorname{Spec}(\mathbb{C}[\Sigma]^H).$$

Apart from the GIT quotient, we can also consider the "stacky" quotient. If we regard stacks as sheaves of groupoids on the big étale site $(Sch/\mathbb{C})_{ét}$, then the *quotient stack* $[\Sigma/H]$ can be understood as the stackyfication of the presheaf sending every $\mathbb{C}$-scheme S to the *action groupoid* $[\Sigma(S), H(S)]$, whose objects are the S-points of Σ and the morphisms between two S-points $x, y \in \Sigma(S)$ are given by the set

$$\operatorname{Hom}_{[\Sigma(S),H(S)]}(x, y) = \{h \in H(S) : y = h \cdot x\}.$$

It follows easily by imposing the descent condition to this presheaf of groupoids that the quotient stack $[\Sigma/H]$ sends a $\mathbb{C}$-scheme S to a groupoid whose objects are

$$\operatorname{Obj}_{[\Sigma/H](S)} = \left\{(E, f) : E \to H \text{ a } H\text{-torsor}, f : E \to \Sigma \ H\text{-equivariant}\right\},$$

and whose morphisms are morphisms of H-torsors commuting with f.

There is a natural morphism from the quotient stack $[\Sigma/H]$ to the GIT quotient $\mathbf{c}_\Sigma$

$$[\Sigma/H] \longrightarrow \mathbf{c}_\Sigma.$$

Since the actions commute, this map is Z-equivariant and thus we can also define a morphism

$$[\Sigma/(H \times Z)] \longrightarrow [\mathbf{c}_\Sigma/Z].$$

We also have a natural map $[\mathbf{c}_\Sigma/Z] \to \mathbb{B}Z = [\operatorname{Spec}\mathbb{C}/Z]$. The generalized Hitchin map is constructed from the sequence

$$[\Sigma/(H\times Z)] \longrightarrow [\mathbf{c}_\Sigma/Z] \longrightarrow \mathbb{B}Z.$$

We start by taking X a smooth projective $\mathbb{C}$-variety. A morphism $X \to \mathbb{B}Z$ is by definition a Z-bundle $L \to X$. We can now consider $\mathscr{M}_L(\Sigma)$ and $\mathscr{B}_L(\Sigma)$ the stacks of sections of the associated stacks $L([\Sigma/H])$ and $L(\mathbf{c}_\Sigma)$ over X, respectively. Equivalently, $\mathscr{M}_L(\Sigma)$ and $\mathscr{B}_L(\Sigma)$ are the the stacks of maps from X to $[\Sigma/(H\times Z)]$ and to $[\mathbf{c}_\Sigma/Z]$, respectively, lying over the map $X \to \mathbb{B}Z$ defined by L.

The resulting morphism

$$\mathscr{M}_L(\Sigma) \longrightarrow \mathscr{B}_L(\Sigma)$$

is the *generalized Hitchin map*.

We study now some well-known examples that fit in this general picture.

Example 2.1.1 (The Hitchin fibration)**.** The Hitchin fibration as conceived originally by Hitchin [Hit87b] is obtained by taking $H = G$ to be any reductive algebraic group over $\mathbb{C}$, $Z = \mathbb{C}^*$ and $\Sigma = \mathfrak{g}$ the Lie algebra of G, endowed with the adjoint action of G and with the homothety action of $\mathbb{C}^*$.

In this case, the Chevalley restriction theorem tells us that

$$\mathbf{c}_\mathfrak{g} = \mathfrak{g} /\!\!/ G \cong \mathfrak{t}/W,$$

where $\mathfrak{t} \subset \mathfrak{g}$ is a maximal Cartan subalgebra, and W is the Weyl group. Therefore, the quotient $\mathbf{c}_\mathfrak{g}$ is just the affine space $\mathbb{A}^r$, where r is the rank of the group, with coordinates given by a set of generators $b_1, \dots, b_r \in \mathbb{C}[\mathfrak{t}]^W$.

If X is a curve, closed points of the stack $\mathscr{M}_L(\Sigma)$, for $L \to \Sigma$ a line bundle, are L-*twisted Higgs bundles*, that is, pairs (E, φ) with $E \to X$ a G-bundle and $\varphi \in H^0(X, E(\mathfrak{g}) \otimes L)$. Usual Higgs bundles over curves as introduced by Hitchin [Hit87a] arise with $L = K_X$ the canonical line bundle of X.

More generally, one can consider twisting by higher rank bundles if one instead takes $Z = GL_n$ and $\Sigma = \mathfrak{C}^n_\mathfrak{g}$ the commuting scheme

$$\mathfrak{C}^n_\mathfrak{g} = \left\{(x_1, \dots, x_n) \in \mathfrak{g} : [x_i, x_j] = 0, \forall i, j = 1, \dots, n\right\},$$

with the natural action of GL_n. This can be defined over a smooth complex projective variety X of any dimension, and if we fix $n = \dim X$ and the twisting bundle $V : X \to \mathbb{B}\,GL_n$ to be the cotangent bundle $V = \Omega^1_X$, be recover the usual notion of Higgs bundles over projective varieties as introduced by Simpson [Sim88].

Example 2.1.2 (The (additive) Hitchin fibration for symmetric pairs)**.** Let us consider now a reductive algebraic group G over $\mathbb{C}$ and a symmetric subgroup $H \subset G$ associated to an involution $\theta \in \mathrm{Aut}_2(G)$. The involution θ induces a decomposition in $\mathfrak{g}$ in the $+1$ and -1 eigenspaces

$$\mathfrak{g} = \mathfrak{g}^\theta \oplus \mathfrak{m}$$

and the adjoint action of G on $\mathfrak{g}$ induces an action of H on $\mathfrak{m}$. We can now consider a Hitchin fibration associated to H, $Z = \mathbb{C}^*$ and $\Sigma = \mathfrak{m}$. This is the *Hitchin fibration for the symmetric pair* (G, H). This fibration has been studied in the thesis of Peón-Nieto [PN13] and in her later work with García-Prada and Ramanan [GPPNR18, GPPN23]. A complete description will be provided in forthcoming work by Hameister and Morrissey [HM, Ham23].

The objects appearing in this situation are sometimes called *Higgs bundles for real groups*. This is because when one considers the Hitchin fibration for the symmetric pair (G, G^θ) over a curve X, the closed points of the stack $\mathscr{M}_{K_X}(\mathfrak{m})$ which satisfy some condition of polystability correspond under the nonabelian Hodge correspondence to representations of $\pi_1(X)$ into the real form $G_\mathbb{R}$ associated to θ. The reader can consult [GP20] for more information about this.

Example 2.1.3 (The Hitchin map for reductive monoids)**.** Consider a semisimple simply-connected group G over $\mathbb{C}$ and M a very flat monoid with $(M^\times)' = G$. The adjoint action of G on itself extends to an action on M. Moreover, the torus $Z = Z^0_{M^\times}$ also acts on M.

We can then consider the generalized Hitchin map corresponding to the sequence

$$[M/(G \times Z)] \longrightarrow [\mathbf{c}_M/Z] \longrightarrow \mathbb{B}Z.$$

Moreover, the abelianization map $\alpha_M : M \to \mathbb{A}_M$ factors through $\mathbf{c}_M$, so we can consider the sequence

$$[M/(G \times Z)] \longrightarrow [\mathbf{c}_M/Z] \longrightarrow [\mathbb{A}_M/Z] \longrightarrow \mathbb{B}Z.$$

Given a smooth complex projective curve X, we can consider $\mathscr{M}(M)$, $\mathscr{B}(M)$ and $\mathscr{A}(M)$ the stacks of maps from X to $[M/(G\times Z)]$, $[\mathbf{c}_M/Z]$ and $[\mathbb{A}_M/Z]$, respectively. We obtain a sequence

$$\mathscr{M}(M) \xrightarrow{h} \mathscr{B}(M) \longrightarrow \mathscr{A}(M) \longrightarrow \mathrm{Bun}_Z(X).$$

The map $h : \mathscr{M}(M) \to \mathscr{B}(M)$ is the *Hitchin map associated to the monoid* M.

Multiplicative G*-Higgs bundles*

Let G be a reductive algebraic group over $\mathbb{C}$ and X a smooth complex projective curve. For any $d \in \mathbb{Z}_+$ we denote by $X_d = X^d/\mathfrak{S}_d$ the d-th symmetric product, so that elements $D \in X_d$ are effective divisors of degree d on X. More generally, given a tuple $\mathbf{d} = (d_1, \dots, d_n)$ of positive integers, we denote $X_\mathbf{d} = X_{d_1} \times \cdots \times X_{d_n}$ and to any element $\mathbf{D} = (D_1, \dots, D_n) \in X_\mathbf{d}$ we can associate the divisor $D = D_1 + \cdots + D_n$. We denote by $|D|$ the support of the divisor D.

Definition 2.1.4. Let $\mathbf{D} \in X_\mathbf{d}$. A *multiplicative* G*-Higgs bundle* with singularities in $\mathbf{D}$ is a pair (E, φ), where $E \to X$ is a G-bundle and φ is a section of the adjoint bundle of groups $E(G)$ defined over $X \setminus |D|$.

We denote by $\mathscr{M}_\mathbf{d}(G)$ the moduli stack of tuples $(\mathbf{D}, E, \varphi)$ with $\mathbf{D} \in X_\mathbf{d}$ and (E, φ) a multiplicative G-Higgs bundle with singularities in $\mathbf{D}$.

Remark 2.1.5. To any tuple $(\mathbf{D}, E, \varphi) \in \mathscr{M}_{\mathbf{d}}(G)$ and any point $x \in |D|$ we can associate an invariant $\mathrm{inv}_x(\varphi) \in X_*(T)_+$, for $T \subset G$ a maximal torus and a fixed choice of dominant Weyl chamber. This invariant is constructed by considering the completion $\mathcal{O}_x$ of the local ring $\mathscr{O}_{X,x}$ of X at x. By picking a local parameter, the ring $\mathcal{O}_x$ is isomorphic to the ring $\mathcal{O}$ of formal power series, and its quotient field $F_x = \mathrm{qf}(\mathcal{O}_x) \cong F$. If we fix two trivializations of E around the formal disc $\mathrm{Spec}(\mathcal{O}_x)$ and restrict φ to $\mathrm{Spec}(F_x)$ we get an element of $G(F_x)$ which is well defined up to the choice of the trivializations, thus, up to the action of $G(\mathcal{O}_x) \times G(\mathcal{O}_x)$ by left and right multiplication. That is, we obtain a well defined element $\lambda_x \in G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x)$. Recall that $G(\mathcal{O})\backslash G(F)/G(\mathcal{O})$ is in natural bijection with $X_*(T)_+$. We thus define

$$\mathrm{inv}_x(\varphi) = \lambda_x \in X_*(T)_+.$$

Globally, if $D = \sum_{x \in X} n_x x$, we get a $X_*(T)_+$-valued divisor

$$\mathrm{inv}(\varphi) = \sum_{x \in X} n_x \,\mathrm{inv}_x(\varphi) x.$$

From the remark above it follows that we can prescribe the value of the invariant $\mathrm{inv}(\varphi)$ in order to obtain a finite-type moduli stack. That is, we define

$$\mathscr{M}_{\mathbf{d},\boldsymbol{\lambda}}(G) = \{(\mathbf{D}, E, \varphi) \in \mathscr{M}_{\mathbf{d}}(G) : \mathrm{inv}(\varphi) = \boldsymbol{\lambda} \cdot \mathbf{D}\}.$$

Here, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is a tuple of dominant cocharacters $\lambda_i \in X_*(T)_+$ and $\boldsymbol{\lambda} \cdot \mathbf{D} = \sum_{i=1}^n \lambda_i D_i$. For any element $(\mathbf{D}, E, \varphi)$ in $\mathscr{M}_{\mathbf{d},\boldsymbol{\lambda}}(G)$, we say that the pair (E, φ) is a multiplicative G-Higgs bundle with *singularity type* $(\mathbf{D}, \boldsymbol{\lambda})$. The order on $X_*(T)_+$ induces an order on $X_*(T)_+$-valued divisors and thus we can also define the bigger stack

$$\mathscr{M}_{\mathbf{d},\bar{\boldsymbol{\lambda}}}(G) = \{(\mathbf{D}, E, \varphi) \in \mathscr{M}_{\mathbf{d}}(G) : \mathrm{inv}(\varphi) \leq \boldsymbol{\lambda} \cdot \mathbf{D}\}.$$

We also remark that there is a restriction for the value of the invariant $\mathrm{inv}(\varphi)$, as explained by Hurtubise and Markman [HM02, Remark 8.1]. Let $\mathbf{D} \in X_{\mathbf{d}}$ be a tuple of divisors with $D = \sum_{x \in X} n_x x$ and consider T^{sc} the maximal torus of the simply-connected group G^{sc} isogenous to G. Recall that that there is a natural inclusion $X_*(T^{sc}) \subset X_*(T)$; in fact, $X_*(T^{sc})$ is the coroot lattice of the root system $\Phi_{\mathfrak{g}}$ and the quotient $X_*(T)/X_*(T^{sc})$ is by definition the fundamental group $\pi_1(G)$ of G.

Proposition 2.1.6. *If* (E, φ) *is a multiplicative* G*-Higgs bundle with singularities in* $\mathbf{D}$*, then*

$$\sum_{x \in X} n_x \,\mathrm{inv}_x(\varphi) \in X_*(T^{sc})_+.$$

Proof. Consider the open cover $\mathcal{U}$ of X defined by taking a small disc U_x around any point $x \in |D|$ of the support of D and $U_0 = X \setminus |D|$. Fix a trivialization τ_x of E over every U_x and consider the 1-cocycle $g \in Z^1(\mathcal{U}, E(G))$ with $g_{x0} = \tau_x^{-1} \circ \varphi$. There is a natural short exact sequence

$$1 \longrightarrow X \times \pi_1(G) \longrightarrow E(G^{sc}) \longrightarrow E(G) \longrightarrow 0,$$

which defines a natural homomorphism $H^1(X, E(G)) \to H^2(X, \pi_1(G))$. Identifying $H^2(X, \pi_1(G))$ with $\pi_1(G)$, the image of (the class of) $\mathfrak{g}$ is clearly equal to the class of $\sum_{x \in X} n_x \operatorname{inv}_x(\varphi)$, but note that $\mathfrak{g}$ defines the trivial G-bundle, so this element must vanish in $\pi_1(G)$. ◻

Remark 2.1.7. Compare the previous proposition with Proposition 3.2.6.

The multiplicative Hitchin map

Let us assume now that we are in the situation in which G is semisimple simply-connected so $\mathbb{C}[G]^G$ is a polynomial algebra and generated by the functions $b_1, \dots, b_r$ with $b_i = \operatorname{tr}(\rho_i)$, where ρ_i is the fundamental representation associated to the fundamental dominant weight $\omega_i \in X^*(T)_+$. Consider now the bundle $\mathscr{B}_{\mathbf{d},\boldsymbol{\lambda}}(G) \to X_{\mathbf{d}}$ whose fibre over $\mathbf{D}$ is the space of sections

$$\mathscr{B}_{\mathbf{d},\boldsymbol{\lambda}}(G)_{\mathbf{D}} = \bigoplus_{i=1}^{r} H^0(X, \mathscr{O}_X(\langle \omega_i, \boldsymbol{\lambda} \cdot \mathbf{D} \rangle)).$$

Definition 2.1.8. The *Hitchin map* for multiplicative G-Higgs bundles is the morphism

$$\begin{aligned} h_{\mathbf{d},\boldsymbol{\lambda}} : \mathscr{M}_{\mathbf{d},\boldsymbol{\lambda}}(G) &\longrightarrow \mathscr{B}_{\mathbf{d},\boldsymbol{\lambda}}(G) \\ (\mathbf{D}, E, \varphi) &\longmapsto (b_1(\varphi), \dots, b_r(\varphi)). \end{aligned}$$

In the following sections, we will be mainly concerned with the simpler case where the d_i are equal to 1 and thus the divisors D_i consist of single points, so we have that $\mathbf{D} = \mathbf{x} = (x_1, \dots, x_n)$ is just a tuple of n different points of X. In that case, denoting $\vec{1} = (1, \dots, 1)$, we define

$$\mathscr{M}_{\mathbf{x},\boldsymbol{\lambda}}(G) := \mathscr{M}_{\vec{1},\boldsymbol{\lambda}}(G)_{\mathbf{x}} = \left\{(E, \varphi) \in \mathscr{M}_{\vec{1}}(G) : \operatorname{inv}(\varphi) = \boldsymbol{\lambda} \cdot \mathbf{x}\right\}.$$

We can also consider the bigger stack

$$\mathscr{M}_{\mathbf{x},\bar{\boldsymbol{\lambda}}}(G) := \mathscr{M}_{\vec{1},\bar{\boldsymbol{\lambda}}}(G)_{\mathbf{x}} = \left\{(E, \varphi) \in \mathscr{M}_{\vec{1}}(G) : \operatorname{inv}(\varphi) \leq \boldsymbol{\lambda} \cdot \mathbf{x}\right\}.$$

When G is semisimple simply-connected, we can also consider the Hitchin base

$$\mathscr{B}_{\mathbf{x},\boldsymbol{\lambda}}(G) = \mathscr{B}_{\vec{1},\boldsymbol{\lambda}}(G)_{\mathbf{x}} = \bigoplus_{i=1}^{r} H^0(X, \mathcal{O}_X(\langle \omega_i, \boldsymbol{\lambda} \cdot \mathbf{x} \rangle)),$$

and the Hitchin map

$$\begin{aligned} h_{\mathbf{x},\boldsymbol{\lambda}} : \mathscr{M}_{\mathbf{x},\boldsymbol{\lambda}}(G) &\longrightarrow \mathscr{B}_{\mathbf{x},\boldsymbol{\lambda}}(G) \\ (E, \varphi) &\longmapsto (b_1(\varphi), \dots, b_r(\varphi)). \end{aligned}$$

The multiplicative Hitchin map can be recovered as a pullback of the Hitchin map for monoids as follows. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be a tuple of dominant cocharacters $\lambda_i \in X_*(T)_+$. The natural projection $T \to T^{ad}$ induces a projection $X_*(T)_+ \to X_*(T^{ad})_+$. Therefore $\boldsymbol{\lambda}$ defines a multiplicative map

$$\begin{aligned} \boldsymbol{\lambda} : (\mathbb{C}^*)^n &\longrightarrow T^{ad} \\ (z_1, \dots, z_n) &\longmapsto z_1^{\lambda_1} \cdots z_n^{\lambda_n} \end{aligned}$$

which extends to a map $\boldsymbol{\lambda} : \mathbb{A}^n \to \mathbb{A}_{\mathrm{Env}(G)}$. Indeed, recall that $P_+(\mathrm{Env}(G)) = \mathbb{Z}_+\langle \Delta_{\mathfrak{g}} \rangle$, so $P_+(\mathrm{Env}(G))^\vee = X_*(T^{ad})_+$. We can now define a very flat monoid $M^{\boldsymbol{\lambda}}$ such that the derived group of its unit group is equal to G through the Cartesian diagram

$$\begin{array}{ccc} M^{\boldsymbol{\lambda}} & \longrightarrow & \mathrm{Env}(G) \\ \downarrow & & \downarrow{\scriptstyle \alpha_{\mathrm{Env}(G)}} \\ \mathbb{A}^n & \xrightarrow{-w_0\boldsymbol{\lambda}} & \mathbb{A}_{\mathrm{Env}(G)}. \end{array}$$

By construction we get that the connected centre of the unit group of $M^{\boldsymbol{\lambda}}$ is $Z = (\mathbb{C}^*)^n$ and $\mathbb{A}_{M^{\boldsymbol{\lambda}}} = \mathbb{A}^n$, so $\mathrm{Bun}_Z(X) = \mathrm{Pic}(X)^n$ and, for any tuple of line bundles $\mathbf{L} = (L_1, \dots, L_n)$, the fibre of the natural map $\mathscr{A}(M^{\boldsymbol{\lambda}}) \to \mathrm{Bun}_Z(X)$ over $\mathbf{L}$ is $\mathscr{A}_{\mathbf{L}}(M) = \bigoplus_{i=1}^n H^0(X, L_i)$. If we replace $\mathrm{Env}(G)$ by $\mathrm{Env}^0(G)$ we obtain an open dense subvariety $M^{\boldsymbol{\lambda},0} \subset M^{\boldsymbol{\lambda}}$, and we can also consider the stack $\mathscr{M}(M^{\boldsymbol{\lambda},0})$ of maps from X to $[M^{\boldsymbol{\lambda},0}/(G \times Z)]$ which is naturally an open substack of $\mathscr{M}(M^{\boldsymbol{\lambda}})$.

Consider now $\mathbf{d} = (d_1, \dots, d_n)$ a tuple of positive integers and an element $\mathbf{D} = (D_1, \dots, D_n) \in X_{\mathbf{d}}$. To each D_i we can associate the line bundle $\mathcal{O}_X(D_i)$ which, since D_i is efective has a canonical non-vanishing section $s_i \in H^0(X, \mathcal{O}_X(D_i))$. Let us denote $\mathcal{O}_X(\mathbf{D}) = \bigoplus_{i=1}^n \mathcal{O}_X(D_i)$ and $\mathbf{s} = (s_1, \dots, s_n)$. The following result is essentially contained in the works of Bouthier, J. Chi and G. Wang [Bou15, BC18, Bou17, Chi22, Wan23], and it also follows from our more general result Theorem 2.2.5.

Theorem 2.1.9 (Bouthier–Chi–Wang)**.** *The map*

$$\begin{aligned} X_{\mathbf{d}} &\longrightarrow \mathscr{A}(M^{\boldsymbol{\lambda}}) \\ \mathbf{D} &\longmapsto (\mathcal{O}_X(\mathbf{D}), \mathbf{s}) \end{aligned}$$

induces the following diagram, where all squares are Cartesian,

$$\begin{array}{ccccccc} \mathscr{M}_{\mathbf{d},\boldsymbol{\lambda}}(G) & \hookrightarrow & \mathscr{M}_{\mathbf{d},\bar{\boldsymbol{\lambda}}}(G) & \xrightarrow{h_{\mathbf{d},\boldsymbol{\lambda}}} & \mathscr{B}_{\mathbf{d},\boldsymbol{\lambda}}(G) & \longrightarrow & X_{\mathbf{d}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathscr{M}(M^{\boldsymbol{\lambda},0}) & \hookrightarrow & \mathscr{M}(M^{\boldsymbol{\lambda}}) & \xrightarrow{h_X} & \mathscr{B}(M^{\boldsymbol{\lambda}}) & \longrightarrow & \mathscr{A}(M^{\boldsymbol{\lambda}}). \end{array}$$

The moduli space of simple pairs

Let G be a reductive algebraic group over $\mathbb{C}$. Apart from the moduli stack $\mathscr{M}_{\mathbf{d},\boldsymbol{\lambda}}(G)_{\mathbf{D}}$, which classifies multiplicative G-Higgs bundles with singularity type

$(\mathbf{D}, \boldsymbol{\lambda})$, after an argument by Hurtubise and Markman [HM02], one can consider a *moduli space* for *simple* multiplicative Higgs bundles.

We begin by noting that, if (E, φ) is a multiplicative G-Higgs bundle, the section φ of $E(G)$ can be regarded as an automorphism $\varphi : E \to E$ defined over X' the complement of some finite subset of X. A *morphism* of multiplicative G-Higgs bundles $(E_1, \varphi_1) \to (E_2, \varphi_2)$ can be then regarded as a map $E_1 \to E_2$ such that the following diagram commutes

$$\begin{array}{ccc} E_1|_{X'} & \longrightarrow & E_2|_{X'} \\ \big\downarrow{\scriptstyle \varphi_1} & & \big\downarrow{\scriptstyle \varphi_2} \\ E_1|_{X'} & \longrightarrow & E_2|_{X'}. \end{array}$$

Since the centre $Z_G \subset G$ acts trivially on $E(G)$, automorphisms of E of the form $e \mapsto e \cdot z$ for $z \in Z_G$ give automorphisms of any multiplicative G-Higgs bundle of the form (E, φ). This gives a natural inclusion $Z_G \hookrightarrow \mathrm{Aut}(E, \varphi)$.

Definition 2.1.10. A multiplicative G-Higgs bundle (E, φ) is *simple* if the natural inclusion $Z_G \hookrightarrow \mathrm{Aut}(E, \varphi)$ is a bijection.

The argument for the existence of a moduli space of simple multiplicative Higgs bundles goes as follows. Let $\rho : G \to GL(V)$ be a faithful representation of G and, for $T \subset G$ a maximal torus, let

$$P(V) = \{\chi \in X^*(T) : \exists v \in V \setminus \{0\} \text{ such that } \rho(t)v = t^\chi v, \forall t \in T\}$$

be its weight lattice. For any cocharacter $\lambda \in X_*(T)$ we can define the number

$$d_\rho(\lambda) = \min\left\{\langle w\lambda, \chi\rangle : w \in W_T, \chi \in P(V)\right\},$$

for W_T the Weyl group of T. Let us fix a Borel subgroup B of G containing T. If $\lambda \in X_*(T)_+$, for any $\phi \in G(F)$ that lies in the orbit $G(\mathcal{O})z^\lambda G(\mathcal{O})$, the highest pole order in the coefficients of the matrix $\rho(\phi) \in \mathrm{End}\, V \otimes F$ is equal to $d_\rho(\lambda)$.

Let $\mathbf{D} = (D_1, \dots, D_n)$ be a tuple of effective divisors $D_i \in X_{d_i}$, let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be a tuple of dominant cocharacters $\lambda_i \in X_*(T)_+$, and consider the $X_*(T)_+$-valued divisor $\boldsymbol{\lambda} \cdot \mathbf{D} = \sum_{i=1}^n \lambda_i D_i$. Define now the divisor

$$(\boldsymbol{\lambda} \cdot \mathbf{D})_\rho := -\sum_{i=1}^n d(\lambda_i) D_i.$$

Then, for any multiplicative G-Higgs bundle (E, φ) with $\mathrm{inv}(\varphi) \leq \boldsymbol{\lambda} \cdot \mathbf{D}$, the representation ρ induces a section

$$\rho(\varphi) \in H^0(X \setminus |D|, \mathrm{End}\, E(V) \otimes \mathcal{O}_X((\boldsymbol{\lambda} \cdot \mathbf{D})_\rho)).$$

Indeed, if $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$, then $d_\rho(\lambda) \leq d_\rho(\mu)$, so $(\boldsymbol{\lambda} \cdot \mathbf{D})_\rho \geq (\boldsymbol{\mu} \cdot \mathbf{D})_\rho$.

A pair (E, φ), where E is a G-bundle on X and φ is a section of $\mathrm{End}\, E(V) \otimes L$ for some representation $\rho : G \to GL(V)$ and for some line bundle $L \to X$, is called an

L-twisted ρ-Higgs pair. We denote the moduli stack of these pairs by $\mathrm{Higgs}_{L,\rho}(G)$. What we have just shown is that ρ induces an inclusion

$$\mathscr{M}_{d,\bar{\lambda}}(G)_{\mathbf{D}} \longrightarrow \mathrm{Higgs}_{\mathcal{O}_X((\boldsymbol{\lambda}\cdot\mathbf{D})_\rho),\rho}(G).$$

The stability conditions and good moduli space theory for L-twisted ρ-Higgs pairs are well known (the reader may refer to [GPGiR12] or [Sch08] for general treatments of the topic). The existence of moduli spaces of simple L-twisted ρ-Higgs bundles guarantees the existence of good moduli spaces of simple pairs $\mathbf{M}_{\mathbf{D},\bar{\lambda}}(G)$ and $\mathbf{M}_{\mathbf{D},\lambda}(G)$ for $\mathscr{M}_{d,\bar{\lambda}}(G)_{\mathbf{D}}$ and $\mathscr{M}_{d,\lambda}(G)_{\mathbf{D}}$, at least as non-Hausdorff complex spaces. For more details, see Remark 2.5 in [HM02].

The tangent space

Although the moduli space of simple pairs $\mathbf{M}_{\mathbf{D},\lambda}(G)$ might be non-Hausdorff, one can still study its tangent space at some point $[E,\varphi]$, the isomorphism class of a pair (E,φ). Using standard arguments in deformation theory (see, for example [BR94]), Hurtubise and Markman [HM02] showed that the Zariski tangent space of $\mathbf{M}_{\mathbf{D},\lambda}(G)$ at $[E,\varphi]$ is equal to $\mathbb{H}^1(C_{[E,\varphi]})$, the first hypercohomology space of the deformation complex

$$C_{[E,\varphi]}: \qquad E(\mathfrak{g}) \xrightarrow{\mathrm{ad}_\varphi} \mathrm{ad}(E,\varphi).$$

Here, $\mathrm{ad}(E,\varphi)$ is a vector bundle on X defined by the following short exact sequence

$$0 \longrightarrow \{(a,b) : a + \mathrm{Ad}_\varphi(b) = 0\} \hookrightarrow E(\mathfrak{g}) \oplus E(\mathfrak{g}) \longrightarrow \mathrm{ad}(E,\varphi) \longrightarrow 0,$$

where, if we regard $b \in E(\mathfrak{g})$ and $\varphi \in E(G)$, as maps $f_b : E \to \mathfrak{g}$, $f_\varphi : E \to G$, by $\mathrm{Ad}_\varphi(b)$ we mean the map sending any $e \in E$ to $\mathrm{Ad}_{f_\varphi(e)}(f_b(e))$. The map ad_φ is defined as

$$\mathrm{ad}_\varphi = L_\varphi - R_\varphi,$$

where L_φ and R_φ are given by the following diagram

$$\begin{array}{ccc} E(\mathfrak{g}) & & \\ \big\downarrow i_1 & \searrow^{L_\varphi} & \\ E(\mathfrak{g}) \oplus E(\mathfrak{g}) & \longrightarrow & \mathrm{ad}(E,\varphi) \\ \big\uparrow i_2 & \nearrow_{R_\varphi} & \\ E(\mathfrak{g}). & & \end{array}$$

Remark 2.1.11. In terms of a faithful representation $\rho : G \hookrightarrow GL(V)$, the maps L_φ and R_φ correspond precisely to left and right multiplication by $\rho(\varphi)$, and the deformation complex becomes

$$\mathrm{ad}_{\rho(\varphi)} : \mathrm{End}\, E(V_\rho) \longrightarrow \mathrm{End}\, E(V_\rho) \otimes \mathcal{O}_X((\boldsymbol{\lambda}\cdot\mathbf{D})_\rho)$$
$$\psi \longmapsto [\rho(\varphi), \psi].$$

This is precisely the deformation complex for $\mathcal{O}_X((\boldsymbol{\lambda}\cdot\mathbf{D})_\rho)$-twisted Higgs bundles.

We can also give a explicit description of the tangent space $\mathbb{H}^1(C_{[E,\varphi]})$. We do this by taking an acyclic resolution of the complex $C_{[E,\varphi]}$ and computing cohomology of the total complex. For example, we can take the Čech resolution associated to an acyclic étale cover of X, that we denote by $\{U_i\}_i$. We can now compute $\mathbb{H}^1(C_{[E,\varphi]})$ as the quotient Z/B, where Z consists of pairs $(\mathbf{s}, \mathbf{t})$, with $\mathbf{s} = (s_{ij})_{i,j}$, $\mathbf{t} = (t_i)_i$, for the $s_{ij} \in \Gamma(U_i \cap U_j, E(\mathfrak{g}))$, and $t_i \in \Gamma(U_i, \mathrm{ad}(E, \varphi))$, satisfying the equations

$$\begin{cases} s_{ij} + s_{jk} = s_{ij}, \\ t_i - t_j = \mathrm{ad}_\varphi(s_{ij}); \end{cases}$$

and B is the set of pairs $(\mathbf{s}, \mathbf{t})$ of the form $\mathbf{s} = (r_i - r_j)_{i,j}$ and $\mathbf{t} = (\mathrm{ad}_\varphi(r_i))_i$, for some $\mathbf{r} = (r_i)_i$, with $r_i \in \Gamma(U_i, E(\mathfrak{g}))$. Now one obtains a deformation of the pair (E, φ) from a pair $(\mathbf{s}, \mathbf{t})$ by considering the pair $(E, \varphi)_{(\mathbf{s},\mathbf{t})}$ over $X \times \mathrm{Spec}(\mathbb{C}[\delta])$, for $\mathbb{C}[\delta] = \mathbb{C}[t]/(t^2)$, determined by

$$\begin{cases} \mathbf{g}^{\mathbf{s}} = \mathbf{g}(1 + \delta \mathbf{s}), \\ \boldsymbol{\phi}^{\mathbf{t}} = \boldsymbol{\phi}(1 + \delta \mathbf{t}). \end{cases}$$

Here, $\mathbf{g} = (g_{ij})_{i,j}$ are the transition functions of E and $\boldsymbol{\phi} = (\phi_i)_i$ is determined by restricting φ to the U_i.

The symplectic structure

Going on with the study of the deformation theory of multiplicative Higgs bundles, one can show that when the curve X has genus 1 (i.e., if it is an elliptic curve over $\mathbb{C}$) the moduli space admits a symplectic structure. This is a result of Hurtubise and Markman [HM02, Theorem 2.2], and we recall their argument here.

Consider the dual complex of the deformation complex considered above

$$C^*_{[E,\varphi]}: \qquad \mathrm{ad}(E, \varphi)^* \xrightarrow{\mathrm{ad}^*_\varphi} E(\mathfrak{g}^*).$$

Grothendieck–Serre duality gives a perfect pairing

$$\mathbb{H}^1(C_{[E,\varphi]}) \times \mathbb{H}^1(C^*_{[E,\varphi]} \otimes K_X) \longrightarrow H^1(X, K_X) \cong \mathbb{C}.$$

If we assume that $K_X = \mathcal{O}_X$ (which for a smooth projective curve only happens when its genus is 1), we get a perfect pairing between $\mathbb{H}^1(C_{[E,\varphi]})$ and $\mathbb{H}^1(C^*_{[E,\varphi]})$, so we can identify the cotangent space of the moduli space at $[E, \varphi]$ with $\mathbb{H}^1(C^*_{[E,\varphi]})$.

An invariant bilinear form on $\mathfrak{g}$ gives a natural isomorphism between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. Under this isomorphism, the complex $C^*_{[E,\varphi]}$ can be identified with

$$C^*_{[E,\varphi]}: \qquad \mathrm{ad}(E, \varphi^{-1}) \xrightarrow{-\mathrm{ad}_\varphi} E(\mathfrak{g}).$$

We can then give a explicit description of the cotangent space $\mathbb{H}^1(C^*_{[E,\varphi]})$ as the quotient Z^*/B^*, where Z^* consists of pairs $(\boldsymbol{\sigma}, \boldsymbol{\tau})$, with $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j}$, $\boldsymbol{\tau} = (\tau_i)_i$, $\sigma_{ij} \in \Gamma(U_i \cap U_j, \mathrm{ad}(E, \varphi^{-1}))$, $\tau_i \in \Gamma(U_i, E(\mathfrak{g}))$, satisfying

$$\begin{cases} \sigma_{ij} + \sigma_{jk} = \sigma_{ij}, \\ \tau_i - \tau_j = -\mathrm{ad}_\varphi(\sigma_{ij}); \end{cases}$$

and B^* is the set of pairs $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ of the form $\boldsymbol{\sigma} = (\eta_i - \eta_j)_{i,j}$ and $\boldsymbol{\tau} = (-\mathrm{ad}_\varphi(\eta_i))_i$, for some $\boldsymbol{\eta} = (\eta_i)_i$, with $\eta_i \in \Gamma(U_i, \mathrm{ad}(E, \varphi^{-1}))$. One can now check that, under this description, the Grothendieck–Serre duality pairing is given explicitly by

$$\begin{aligned} \Phi : \mathbb{H}^1(C_{[E,\varphi]}) \times \mathbb{H}^1(C^*_{[E,\varphi]} \otimes K_X) &\longrightarrow H^1(X, K_X) \cong \mathbb{C} \\ ([\mathbf{s}, \mathbf{t}], [\boldsymbol{\sigma}, \boldsymbol{\tau}]) &\longmapsto \Phi((\mathbf{s}, \mathbf{t}), (\boldsymbol{\sigma}, \boldsymbol{\tau})) = \langle \mathbf{s}, \boldsymbol{\tau} \rangle - \langle \boldsymbol{\sigma}, \mathbf{t} \rangle, \end{aligned}$$

where $\langle -, - \rangle$ denotes the natural duality pairing.

Finally, note that we can define a morphism of complexes $\Psi : C_{[E,\varphi]} \to C^*_{[E,\varphi]}$, as in the following diagram

$$\begin{array}{ccccc} C_{[E,\varphi]} : & & E(\mathfrak{g}) & \xrightarrow{\mathrm{ad}_\varphi} & \mathrm{ad}(E, \varphi) \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle -L_{\varphi^{-1}}} & & \downarrow{\scriptstyle L_{\varphi^{-1}}} \\ C^*_{[E,\varphi]} : & & \mathrm{ad}(E, \varphi^{-1}) & \xrightarrow{-\mathrm{ad}_\varphi} & E(\mathfrak{g}). \end{array}$$

The adjoint of this morphism is given by

$$\begin{array}{ccccc} C_{[E,\varphi]} : & & E(\mathfrak{g}) & \xrightarrow{\mathrm{ad}_\varphi} & \mathrm{ad}(E, \varphi) \\ \downarrow{\scriptstyle \Psi^\dagger} & & \downarrow{\scriptstyle -R_{\varphi^{-1}}} & & \downarrow{\scriptstyle R_{\varphi^{-1}}} \\ C^*_{[E,\varphi]} : & & \mathrm{ad}(E, \varphi^{-1}) & \xrightarrow{-\mathrm{ad}_\varphi} & E(\mathfrak{g}). \end{array}$$

Now, these two maps are homotopic, since the following diagram commutes

$$\begin{array}{ccc} E(\mathfrak{g}) & \xrightarrow{\mathrm{ad}_\varphi} & \mathrm{ad}(E, \varphi) \\ \downarrow{\scriptstyle -\mathrm{ad}_{\varphi^{-1}}} & \swarrow{\scriptstyle h} & \downarrow{\scriptstyle \mathrm{ad}_{\varphi^{-1}}} \\ \mathrm{ad}(E, \varphi^{-1}) & \xrightarrow{-\mathrm{ad}_\varphi} & E(\mathfrak{g}), \end{array}$$

for $h = L_{\varphi^{-1}} \circ R_{\varphi^{-1}}$. Therefore, Ψ and $\Psi^\dagger$ define the same map in hypercohomology. Moreover, this map is an isomorphism. We can now define the *Hurtubise–Markman symplectic form* as

$$\begin{aligned} \Omega : \mathbb{H}^1(C_{[E,\varphi]}) \times \mathbb{H}^1(C_{[E,\varphi]}) &\longrightarrow \mathbb{C} \\ (\boldsymbol{v}, \boldsymbol{w}) &\longmapsto \Phi(\boldsymbol{v}, \Psi(\boldsymbol{w})). \end{aligned}$$

This is obviously non-degenerate and it is clearly a 2-form since

$$\begin{aligned} \Omega((\mathbf{s}, \mathbf{t}), (\mathbf{s}', \mathbf{t}')) &= \langle \mathbf{s}, \Psi(\mathbf{t}') \rangle - \langle \Psi(\mathbf{s}'), \mathbf{t} \rangle = \langle \Psi^\dagger(\mathbf{s}), \mathbf{t}' \rangle - \langle \mathbf{s}', \Psi^\dagger(\mathbf{t}) \rangle \\ &= \langle \Psi(\mathbf{s}), \mathbf{t}' \rangle - \langle \mathbf{s}', \Psi(\mathbf{t}) \rangle = -\Omega((\mathbf{s}', \mathbf{t}'), (\mathbf{s}, \mathbf{t})). \end{aligned}$$

Hurtubise and Markman [HM02] show that it is closed.

2.2 THE HITCHIN MAP FOR SYMMETRIC VARIETIES

Our purpose in this section is to give a generalization of the multiplicative Hitchin map to the more general situation of a pair formed by a reductive group and an involution on it. Equivalently, this can be understood as a "multiplicative analog" of the Hitchin fibration for symmetric pairs.

The multiplicative Hitchin map for symmetric embeddings

We begin by taking a semisimple simply-connected group G over $\mathbb{C}$ and an involution $\theta \in \mathrm{Aut}_2(G)$. Let Σ be a very flat symmetric embedding with $O'_\Sigma = G/G^\theta$.

The left multiplication action of G^θ on G/G^θ extends to an action of G^θ on Σ. Moreover, if O_Σ is a symmetric variety the form G_Z/H_Z for some torus Z, then the torus $Z_\Sigma = Z/Z_2$ also acts on Σ through the action of Z.

We can now consider the quotient stacks $[\Sigma/G^\theta]$ and $[\Sigma/(G^\theta \times Z_\Sigma)]$. The quotient map $\Sigma \to \mathbf{c}_\Sigma = \Sigma /\!\!/ G^\theta$ induces a natural map $[\Sigma/G^\theta] \to \mathbf{c}_\Sigma$. We can also descend this to a map $[\Sigma/(G^\theta \times Z_\Sigma)] \to [\mathbf{c}_\Sigma/Z_\Sigma]$.

From the description of $\mathbf{c}_\Sigma$ that we gave in the previous section, it follows that the abelianization map $\alpha_\Sigma : \Sigma \to \mathbb{A}_\Sigma = \Sigma /\!\!/ G$ factors through $\Sigma \to \mathbf{c}_\Sigma \to \mathbb{A}_\Sigma$. The torus Z_Σ clearly acts on the abelianization $\mathbb{A}_\Sigma$, so composing with the natural map $\mathbb{A}_\Sigma \to \operatorname{Spec}\mathbb{C}$, we obtain a sequence of stacks

$$\left[\Sigma/(G^\theta \times Z_\Sigma)\right] \longrightarrow [\mathbf{c}_\Sigma/Z_\Sigma] \longrightarrow [\mathbb{A}_\Sigma/Z_\Sigma] \longrightarrow \mathbb{B}Z_\Sigma.$$

Let X be a smooth complex projective curve. Let $\mathscr{M}(\Sigma)$, $\mathscr{B}(\Sigma)$ and $\mathscr{A}(\Sigma)$ denote the stacks of maps from X to $[\Sigma/(G^\theta \times Z_\Sigma)]$, $[\mathbf{c}_\Sigma/Z_\Sigma]$ and $[\mathbb{A}_\Sigma/Z_\Sigma]$, respectively. We obtain a sequence

$$\mathscr{M}(\Sigma) \xrightarrow{\;h\;} \mathscr{B}(\Sigma) \longrightarrow \mathscr{A}(\Sigma) \longrightarrow \mathrm{Bun}_{Z_\Sigma}(X).$$

Definition 2.2.1. The map $h : \mathscr{M}(\Sigma) \to \mathscr{B}(\Sigma)$ is the *Hitchin map associated to the symmetric embedding* Σ.

If we let $L \to X$ be a Z_Σ-bundle, the natural map $X \to \mathbb{B}Z_\Sigma$ associated to L induces the following diagram, where all squares are Cartesian

$$\begin{array}{ccccccc}
L(\left[\Sigma/G^\theta\right]) & \longrightarrow & L(\mathbf{c}_\Sigma) & \longrightarrow & L(\mathbb{A}_\Sigma) & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle L} \\
\left[\Sigma/(G^\theta \times Z_\Sigma)\right] & \longrightarrow & [\mathbf{c}_\Sigma/Z_\Sigma] & \longrightarrow & [\mathbb{A}_\Sigma/Z_\Sigma] & \longrightarrow & \mathbb{B}Z_\Sigma.
\end{array}$$

These $L([\Sigma/G^\theta])$, $L(\mathbf{c}_\Sigma)$ and $L(\mathbb{A}_\Sigma)$ are naturally stacks over X and we denote by $\mathscr{M}_L(\Sigma)$, $\mathscr{B}_L(\Sigma)$ and $\mathscr{A}_L(\Sigma)$ their stacks of sections. We obtain a sequence

$$\mathscr{M}_L(\Sigma) \xrightarrow{\;h_L\;} \mathscr{B}_L(\Sigma) \longrightarrow \mathscr{A}_L(\Sigma).$$

This sequence is the fibre over $L \in \mathrm{Bun}_{Z_\Sigma}(X)$ of the sequence defined above. Fixing a section s in $\mathscr{A}_L(\Sigma)$, over it we can also obtain a morphism of stacks

$$\mathscr{M}_s(\Sigma) \xrightarrow{h_s} \mathscr{B}_s(\Sigma).$$

We can give a more explicit description of all the objects taking part in this definition. The bundle $L(\mathbb{A}_\Sigma)$ is just the associated bundle over X defined by the action of Z_Σ on the A_Σ-toric variety $\mathbb{A}_\Sigma$. Since $\mathbb{A}_\Sigma$ is an affine space, $L(\mathbb{A}_\Sigma)$ is a vector bundle of rank equal to the rank of A_Σ. The bundle $L(\mathbf{c}_\Sigma)$ is also an associated bundle, this time to the action of Z_Σ on $\mathbf{c}_\Sigma$ and, since $\mathbf{c}_\Sigma$ is also an affine space, $L(\mathbf{c}_\Sigma)$ is also a vector bundle, of rank equal to the rank of A_Σ plus the rank of G/G^θ. The stacks $\mathscr{A}_L(\Sigma) = H^0(X, L(\mathbb{A}_\Sigma))$ and $\mathscr{B}_L(\Sigma) = H^0(X, L(\mathbf{c}_\Sigma))$ are just the spaces of sections of these vector bundles.

More precisely, $\mathbb{A}_\Sigma$ is a A_Σ-toric variety with weight semigroup $P_+(\mathbb{A}_\Sigma) \subset X^*(A_\Sigma) \hookrightarrow X^*(Z_\Sigma)$. If we take $\gamma_1, \ldots, \gamma_s \in X^*(Z_\Sigma)$ to be generators of $P_+(\mathbb{A}_\Sigma)$, then $L(\mathbb{A}_\Sigma) = \bigoplus_{i=1}^s L_{\gamma_i}$, for L_{γ_i} the associated line bundle to the action of Z_Σ on $\mathbb{C}^*$ defined by γ_i. On the other hand, $\mathbb{C}[\mathbf{c}_\Sigma]$ is generated by the same e^{γ_i} and by some functions $b_i \in \mathbb{C}[G/G^\theta]$ with weight ϖ_i, so, $L(\mathbf{c}_\Sigma) = L(\mathbb{A}_\Sigma) \oplus \bigoplus_{i=1}^l (\psi^* L)_{\varpi_i}$. Here, $\psi^* L$ is the A_{G^θ}-bundle on X obtained as the image of L through the map $\mathbb{B}Z_\Sigma \to \mathbb{B}A_{G^\theta}$ induced by $\psi : X^*(A_{G^\theta}) \to X^*(Z_\Sigma)$.

Now, a morphism $X \to [\Sigma/G]$ consists of a pair (E, φ), where $E \to X$ is a G^θ-bundle and $\varphi \in H^0(X, E(\Sigma))$ is a section of the associated bundle $E(\Sigma)$ defined by the action of G^θ on Σ. Such a pair is called a *mutiplicative Σ-Higgs pair*.

Now the sequence $\mathscr{M}_L(\Sigma) \to \mathscr{B}_L(\Sigma) \to \mathscr{A}_L(\Sigma)$ defining the multiplicative Hitchin map can be explicitly described as

$$(E, \varphi) \longmapsto (b(\varphi), \alpha_\Sigma(\varphi)) \longmapsto \alpha_\Sigma(\varphi),$$

for $b(\varphi) = (b_1(\varphi), \ldots, b_l(\varphi))$.

Multiplicative (G, θ)*-Higgs bundles*

Let now G be any reductive algebraic group over $\mathbb{C}$ and $\theta \in \mathrm{Aut}_2(G)$ an involution. As above, we let X be a smooth complex projective curve. We can now define multiplicative Higgs bundles associated to the reductive group with involution (G, θ) in a very similar way as how we defined multiplicative G-Higgs bundles in Section 2.1. We reuse our notations from there.

Definition 2.2.2. Let $\mathbf{D} \in X_{\mathbf{d}}$. A *multiplicative* (G, θ)*-Higgs bundle* with singularities in $\mathbf{D}$ is a pair (E, φ), where $E \to X$ is a G^θ-bundle and φ is a section of the associated bundle of symmetric varieties $E(G/G^\theta)$ defined over $X \setminus |D|$.

We denote by $\mathscr{M}_{\mathbf{d}}(G, \theta)$ the moduli stack of tuples $(\mathbf{D}, E, \varphi)$ with $\mathbf{D} \in X_{\mathbf{d}}$ and (E, φ) a multiplicative (G, θ)-Higgs bundle with singularities in $\mathbf{D}$.

Remark 2.2.3. To any tuple $(\mathbf{D}, E, \varphi) \in \mathscr{M}_{\mathbf{d}}(G, \theta)$ and any point $x \in |D|$ we can associate an invariant $\mathrm{inv}_x(\varphi) \in X_*(A_{G^\theta})_-$, for $A \subset G$ a maximal θ-split torus and a fixed choice of anti-dominant Weyl chamber, in the same way as we did for multiplicative Higgs bundles. We construct this by fixing a trivialization of

E around the formal disc $\mathrm{Spec}(\mathcal{O}_x)$ and restricting φ to $\mathrm{Spec}(F_x)$. This gives an element of $(G/G^\theta)(F)$ well defined up to the action of $G(\mathcal{O})$ given by the choice of trivialization. Thus we obtain a well defined element

$$\mathrm{inv}_x(\varphi) \in (G/G^\theta)(F)/G(\mathcal{O}) \cong X_*(A_{G^\theta})_-.$$

Globally, if $D = \sum_{x\in X} n_x x$, we get a $X_*(A_{G^\theta})_-$-valued divisor

$$\mathrm{inv}(\varphi) = \sum_{x\in X} n_x \,\mathrm{inv}_x(\varphi) x.$$

As we did for multiplicative G-Higgs bundles, we can prescribe the value of the invariant $\mathrm{inv}(\varphi)$ in order to obtain a finite-type moduli stack. We define

$$\mathscr{M}_{\mathbf{d},\boldsymbol{\lambda}}(G,\theta) = \{(\mathbf{D}, E, \varphi) \in \mathscr{M}_{\mathbf{d}}(G,\theta) : \mathrm{inv}(\varphi) = \boldsymbol{\lambda}\cdot\mathbf{D}\},$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ is a tuple of anti-dominant cocharacters $\lambda_i \in X_*(A_{G^\theta})_-$ and $\boldsymbol{\lambda}\cdot\mathbf{D} = \sum_{i=1}^n \lambda_i D_i$. For any element $(\mathbf{D}, E, \varphi)$ in $\mathscr{M}_{\mathbf{d},\boldsymbol{\lambda}}(G,\theta)$, we say that the pair (E, φ) is a multiplicative (G,θ)-Higgs bundle with *singularity type* $(\mathbf{D}, \boldsymbol{\lambda})$. The order on $X_*(A_{G_\theta})_-$ induces an order on $X_*(A_{G_\theta})_-$-valued divisors and thus we can also define the bigger stack

$$\mathscr{M}_{\mathbf{d},\bar{\boldsymbol{\lambda}}}(G,\theta) = \{(\mathbf{D}, E, \varphi) \in \mathscr{M}_{\mathbf{d}}(G,\theta) : \mathrm{inv}(\varphi) \le \boldsymbol{\lambda}\cdot\mathbf{D}\}.$$

Suppose now that $\mathbb{C}[G/G^\theta]^{G^\theta}$ is a polynomial algebra and thus it is generated by some functions $b_1, \dots, b_l$ with weights $\varpi_1, \dots, \varpi_l$ respectively, for $\varpi_1, \dots, \varpi_l$ the fundamental dominant weights of the restricted root system Φ_θ. We can consider the bundle $\mathscr{B}_{\mathbf{d},\boldsymbol{\lambda}}(G,\theta) \to X_{\mathbf{d}}$ whose fibre over $\mathbf{D}$ is the space of sections

$$\mathscr{B}_{\mathbf{d},\boldsymbol{\lambda}}(G,\theta)_{\mathbf{D}} = \bigoplus_{i=1}^{l} H^0(X, \mathcal{O}_X(\langle \varpi_i, w_0\boldsymbol{\lambda}\cdot\mathbf{D}\rangle)).$$

Definition 2.2.4. The *Hitchin map* for multiplicative (G,θ)-Higgs bundles is the morphism

$$\begin{aligned} h_{\mathbf{d},\boldsymbol{\lambda}} : \mathscr{M}_{\mathbf{d},\boldsymbol{\lambda}}(G,\theta) &\longrightarrow \mathscr{B}_{\mathbf{d},\boldsymbol{\lambda}}(G,\theta) \\ (\mathbf{D}, E, \varphi) &\longmapsto (b_1(\varphi), \dots, b_l(\varphi)). \end{aligned}$$

As with multiplicative G-Higgs bundles we can also consider the simpler case where the d_i are equal to 1 and thus we have that $\mathbf{D} = \mathbf{x} = (x_1, \dots, x_n)$. We define

$$\mathscr{M}_{\mathbf{x},\boldsymbol{\lambda}}(G,\theta) := \mathscr{M}_{\vec{1},\boldsymbol{\lambda}}(G,\theta)_{\mathbf{x}} = \left\{(E,\varphi) \in \mathscr{M}_{\vec{1}}(G,\theta) : \mathrm{inv}(\varphi) = \boldsymbol{\lambda}\cdot\mathbf{x}\right\},$$

and the bigger stack

$$\mathscr{M}_{\mathbf{x},\bar{\boldsymbol{\lambda}}}(G,\theta) := \mathscr{M}_{\vec{1},\bar{\boldsymbol{\lambda}}}(G,\theta)_{\mathbf{x}} = \left\{(E,\varphi) \in \mathscr{M}_{\vec{1}}(G,\theta) : \mathrm{inv}(\varphi) \le \boldsymbol{\lambda}\cdot\mathbf{x}\right\}.$$

When $\mathbb{C}[G/G^\theta]$ is polynomial, we can also consider the Hitchin base

$$\mathscr{B}_{\mathbf{x},\boldsymbol{\lambda}}(G,\theta) = \mathscr{B}_{\vec{1},\boldsymbol{\lambda}}(G,\theta)_{\mathbf{x}} = \bigoplus_{i=1}^{l} H^0(X, \mathcal{O}_X(\langle \varpi_i, w_0\boldsymbol{\lambda}\cdot\mathbf{x}\rangle)),$$

and the Hitchin map

$$\begin{aligned} h_{\mathbf{x},\boldsymbol{\lambda}} : \mathscr{M}_{\mathbf{x},\boldsymbol{\lambda}}(G,\theta) &\longrightarrow \mathscr{B}_{\mathbf{x},\boldsymbol{\lambda}}(G,\theta) \\ (E, \varphi) &\longmapsto (b_1(\varphi), \dots, b_l(\varphi)). \end{aligned}$$

Relating the two pictures

In the same way as we recovered the multiplicative Hitchin map as a pullback of the Hitchin map for monoids, we can recover the Hitchin map for multiplicative (G, θ)-Higgs bundles from the Hitchin map for symmetric embeddings. We do this in the case that G is semisimple simply-connected and in the more general case in which G is semisimple, not necessarily simply-connected, and $\mathbb{C}[G/G^\theta]^{G^\theta}$ is a polynomial algebra.

Let us suppose that G is simply-connected. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be a tuple of anti-dominant cocharacters $\lambda_i \in X_*(A_{G^\theta})_-$. The natural projection $A_{G^\theta} \to A_{G_\theta}$ induces a projection $X_*(A_{G^\theta})_- \to X_*(A_{G_\theta})_-$ and thus $\boldsymbol{\lambda}$ defines a multiplicative map

$$\begin{aligned} \boldsymbol{\lambda} : (\mathbb{C}^*)^n &\longrightarrow A_{G_\theta} \\ (z_1, \dots, z_n) &\longmapsto z_1^{\lambda_1} \cdots z_n^{\lambda_n} \end{aligned}$$

which extends to a map $\boldsymbol{\lambda} : \mathbb{A}^n \to \mathbb{A}_{\mathrm{Env}(G/G^\theta)}$ since $P_+(\mathrm{Env}(G/G^\theta))^\vee = X_*(A_{G_\theta})_-$. This allows us to define a very flat symmetric embedding $\Sigma^{\boldsymbol{\lambda}}$ with semisimple part $O'_{\Sigma^\lambda} = G/G^\theta$ as the pullback

$$\begin{array}{ccc} \Sigma^{\boldsymbol{\lambda}} & \longrightarrow & \mathrm{Env}(G/G^\theta) \\ \big\downarrow & & \big\downarrow \alpha_{\mathrm{Env}(G/G^\theta)} \\ \mathbb{A}^n & \xrightarrow{-w_0\boldsymbol{\lambda}} & \mathbb{A}_{\mathrm{Env}(G/G^\theta)}. \end{array}$$

Since for any symmetric embedding A_Σ and Z_Σ differ by a finite subgroup, by construction we get $Z_{\Sigma^\lambda} = (\mathbb{C}^*)^n$ and $\mathbb{A}_{\Sigma^\lambda} = \mathbb{A}^n$, so $\mathrm{Bun}_{Z_{\Sigma^\lambda}}(X) = \mathrm{Pic}(X)^n$ and for any $\mathbf{L} \in \mathrm{Pic}(X)^n$ the fibre of $\mathscr{A}(\Sigma^{\boldsymbol{\lambda}}) \to \mathrm{Bun}_{Z_{\Sigma^\lambda}}(X)$ over $\mathbf{L}$ is $\mathscr{A}_{\mathbf{L}}(\Sigma^{\boldsymbol{\lambda}}) = \bigoplus_{i=1}^n H^0(X, L_i)$. Replacing $\mathrm{Env}(G/G^\theta)$ by $\mathrm{Env}^0(G/G^\theta)$ we obtain an open dense subvariety $\Sigma^{\boldsymbol{\lambda},0} \subset \Sigma^{\boldsymbol{\lambda}}$, and we can also consider the open substack $\mathscr{M}(\Sigma^{\boldsymbol{\lambda},0})$ of $\mathscr{M}(\Sigma^{\boldsymbol{\lambda}})$ of maps from X to $[M^{\boldsymbol{\lambda},0}/(G^\theta \times Z_\Sigma)]$.

We provide now the following generalization of Theorem 2.1.9.

Theorem 2.2.5. *The map*

$$\begin{aligned} X_{\mathbf{d}} &\longrightarrow \mathscr{A}(\Sigma^{\boldsymbol{\lambda}}) \\ \mathbf{D} &\longmapsto (\mathscr{O}_X(\mathbf{D}), \mathbf{s}) \end{aligned}$$

induces the following diagram, where all squares are Cartesian,

$$\begin{array}{ccccccc} \mathscr{M}_{\mathbf{d},\boldsymbol{\lambda}}(G,\theta) & \hookrightarrow & \mathscr{M}_{\mathbf{d},\bar{\boldsymbol{\lambda}}}(G,\theta) & \xrightarrow{h_{\mathbf{d},\boldsymbol{\lambda}}} & \mathscr{B}_{\mathbf{d},\boldsymbol{\lambda}}(G,\theta) & \longrightarrow & X_{\mathbf{d}} \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathscr{M}(\Sigma^{\boldsymbol{\lambda},0}) & \hookrightarrow & \mathscr{M}(\Sigma^{\boldsymbol{\lambda}}) & \xrightarrow{h} & \mathscr{B}(\Sigma^{\boldsymbol{\lambda}}) & \longrightarrow & \mathscr{A}(\Sigma^{\boldsymbol{\lambda}}). \end{array}$$

Proof. From the above disussions it is clear that for any $\mathbf{D} \in X_{\mathbf{d}}$ we can identify the fibre $\mathscr{B}_{\mathcal{O}_X(\mathbf{D})}(\Sigma^{\boldsymbol{\lambda}})$ with the space of sections of the bundle

$$\mathscr{O}_X(\mathbf{D}) \oplus \bigoplus_{i=1}^{l} (\mathscr{O}_X(\mathbf{D}) \times_{w_0\boldsymbol{\lambda}} \mathbb{C}^*)_{\varpi_i} = \mathscr{O}_X(\mathbf{D}) \oplus \bigoplus_{i=1}^{l} \mathscr{O}_X(\langle \varpi_i, w_0\boldsymbol{\lambda} \cdot \mathbf{D}\rangle).$$

It is now clear that the rightmost square is Cartesian.

Take any point $x \in |D|$ and take λ_x the coefficient of the divisor $\boldsymbol{\lambda} \cdot \mathbf{D}$ corresponding to x. This λ_x is an anti-dominant cocharacter that can be written as

$$\lambda_x = \mathbf{m}_x \cdot \boldsymbol{\lambda},$$

where $\mathbf{m}_x = (m_{1x}, \dots, m_{nx})$ is a vector with components m_{ix} for $D_i = \sum_x m_{ix} x$. Now, by mapping $z \mapsto z^{\mathbf{m}_x} = (z^{m_{1x}}, \dots, z^{m_{nx}})$ we obtain a morphism $\mathbf{m} : \mathcal{O} \to \mathbb{A}^n$ and the following diagram, with all squares Cartesian

$$\begin{array}{ccccc} \mathrm{Env}^{\lambda_x}(G/G^\theta) & \longrightarrow & \Sigma^{\boldsymbol{\lambda}} & \longrightarrow & \mathrm{Env}(G/G^\theta) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha_{\mathrm{Env}(G/G^\theta)}} \\ \mathrm{Spec}\,\mathcal{O} & \xrightarrow{\ \mathbf{m}\ } & \mathbb{A}^n & \xrightarrow{-w_0\boldsymbol{\lambda}} & \mathbb{A}_{\mathrm{Env}(G/G^\theta)}. \end{array}$$

Now, if we restrict an element $(E, \varphi) \in \mathscr{M}(\Sigma^{\boldsymbol{\lambda}})$ to $\mathrm{Spec}(F_x)$ we obtain an element of $(G/G^\theta)_+(F_x)$ which must lie in $\mathrm{Env}^{\lambda_x}(G/G^\theta)$. Therefore, by Proposition 1.7.7, the image of $\varphi|_{\mathrm{Spec}(F_x)}$ belongs to $\overline{G(\mathcal{O}_x)z^{\lambda_x}}$ and thus $\mathrm{inv}_x(\varphi) \leq \lambda_x$. Moreover, if we change $\mathrm{Env}(G/G^\theta)$ by $\mathrm{Env}^0(G/G^\theta)$ in the diagram above, the element $\varphi|_{\mathrm{Spec}(F_x)}$ lies in $\mathrm{Env}^{\lambda_x,0}(G/G^\theta)$ and therefore its image belongs to $G(\mathcal{O}_x)z^{\lambda_x}$, so $\mathrm{inv}_x(\varphi) = \lambda_x$. □

Let us now treat the case where G is a semisimple group which is not simply-connected. Let $\pi : \hat{G} \to G$ be the simply-connected cover. This map is a central isogeny and thus its kernel is a subgroup of the centre $\pi_1(G) \subset Z_{\hat{G}}$, the fundamental group of G. Moreover, it is a well known result of Steinberg [Ste68, Theorem 9.16] that any involution $\theta \in \mathrm{Aut}_2(G)$ lifts to an involution $\hat{\theta} \in \mathrm{Aut}_2(\hat{G})$ making the following diagram commute

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\ \hat{\theta}\ } & \hat{G} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ G & \xrightarrow{\ \theta\ } & G. \end{array}$$

The following is also well-known.

Lemma 2.2.6. $\pi(\hat{G}^{\hat{\theta}}) = G_0^\theta$.

Proof. The inclusion $\pi(\hat{G}^{\hat{\theta}}) \subset G_0^\theta$ is clear. Indeed, for any $g \in \hat{G}^{\hat{\theta}}$, we have

$$\theta(\pi(g)) = \pi(\hat{\theta}(g)) = \pi(g),$$

so $\pi(\hat{G}^{\hat{\theta}}) \subset G^\theta$ and the inclusion follows since π is continuous and $\hat{G}^{\hat{\theta}}$ is connected.

For the other inclusion, take $k \in G_0^\theta$ and $g \in \hat{G}$ with $\pi(g) = k$. Now,

$$\pi(g\hat{\theta}(g)^{-1}) = k\theta(k)^{-1} = 1.$$

Thus, $g\hat{\theta}(g)^{-1} \in \pi_1(G) \subset Z_{\hat{G}}$, so $g \in \hat{G}_{\hat{\theta}}$. Recall now the decomposition $G_{\hat{\theta}} = F_{\hat{\theta}}\hat{G}^{\hat{\theta}}$, which implies that there exists some $f \in F_{\hat{\theta}}$ and some $g_0 \in \hat{G}^{\hat{\theta}}$ such that $g = fg_0$. Let $k_0 = \pi(g_0) \in G_0^\theta$. Clearly, $\pi(f) \in F^\theta$, so we obtain a decomposition

$$k = \pi(f)k_0,$$

with $k, k_0 \in G_0^\theta$ and $\pi(f) \in F^\theta$. This implies that $\pi(f) = 1$, and thus $k = k_0 = \pi(g_0)$, and $k \in \pi(\hat{G}^{\hat{\theta}})$. □

It follows from the lemma above that we can define an action of G_0^θ on $\hat{G}/\hat{G}^{\hat{\theta}}$ from the left multiplication action of $\hat{G}^{\hat{\theta}}$. Indeed, for any $k \in G_0^\theta$ we just have to take any $h \in \hat{G}^{\hat{\theta}}$ with $\pi(h) = k$ and define $k \cdot (gG^\theta) = hgG^\theta$. This is well defined since, if h' is such that $\pi(h') = k$, then $h' = zh$ for some $z \in G^\theta \cap Z_G$ and thus $h'gG^\theta = hgG^\theta$.

For the following, we need to introduce a new object.

Definition 2.2.7. A *multiplicative* $(G, \theta)_0$*-Higgs bundle* is a pair (E, φ), where $E \to X$ is a G_0^θ-bundle and φ is a section of the associated bundle of symmetric varieties $E(G/G^\theta)$ defined over X' the complement of a finite subset of X.

The exact sequence $1 \to \pi_1(G) \to \hat{G} \to G \to 1$, induces a fibration $\Gamma \hookrightarrow \hat{G}/\hat{G}^{\hat{\theta}} \twoheadrightarrow G/G^\theta$, for some finite group Γ. For any G_0^θ-bundle, we can consider the bundles $E(G/G^\theta)$ and $E(\hat{G}/\hat{G}^{\hat{\theta}})$, associated to the actions of G_0^θ and we obtain a exact sequence of sets with a distinguished element (where H^0 stands for sets of sections)

$$H^0(X', E(\hat{G}/\hat{G}^{\hat{\theta}})) \longrightarrow H^0(X', E(G/G^\theta)) \xrightarrow{\delta} H^1(X', \Gamma).$$

It follows that to any multiplicative $(G, \theta)_0$-Higgs bundle (E, φ) we can associate the invariant $\delta(\varphi) \in H^1(X', \Gamma)$. Now, the section φ comes from a section of $E(\hat{G}/\hat{G}^{\hat{\theta}})$ if and only if $\delta(\varphi) = 1$. Moreover, the map $\pi : \hat{G}^{\hat{\theta}} \to G_0^\theta$ induces a map $\mathrm{Bun}_{\hat{G}^{\hat{\theta}}} \to \mathrm{Bun}_{G_0^\theta}$. We conclude the following.

Proposition 2.2.8. *Any multiplicative* $(G, \theta)_0$*-Higgs bundle with* $\delta(\varphi) = 1$ *is induced from a multiplicative* $(\hat{G}, \hat{\theta})$*-Higgs bundle from the maps* $\mathrm{Bun}_{\hat{G}^{\hat{\theta}}} \to \mathrm{Bun}_{G_0^\theta}$ *and* $\hat{G}/\hat{G}^{\hat{\theta}} \to G/G^\theta$.

The next step is relating multiplicative $(G, \theta)_0$-Higgs bundles with multiplicative (G, θ)-Higgs bundles. We start by considering a (G, θ)-Higgs bundle (E, φ). The decomposition $G^\theta = F^\theta G_0^\theta$ allows us to define an isomorphism $G^\theta \cong F^\theta \ltimes G_0^\theta$, and this induces a factorization of the G^θ-bundle $E \to X$ as

$$\begin{array}{ccc} E & & \\ & \searrow & \\ \Big\downarrow & & Y = E/G_0^{\theta} \\ & \swarrow & \\ X. & & \end{array}$$

Here, $Y \to X$ is a Galois étale cover with Galois group equal to F^{θ}, while $E \to Y$ has the structure of a principal G_0^{θ}-bundle. Moreover, the deck transformation $Y \to Y$ induced by an element $a \in F^{\theta}$ lifts to a map $\tilde{a} : E \to E$ and thus defines a right F^{θ}-action on the G_0^{θ}-bundle $E \to Y$, which clearly gives an Int-twisted F^{θ}-equivariant structure on it. By this we mean that, for any $e \in E$, $a \in F^{\theta}$ and $g \in G_0^{\theta}$, we have

$$\tilde{a}(e) \cdot g = \tilde{a}(e \cdot aga^{-1}).$$

A similar argument allows us to see the bundle $E(G/G^{\theta}) \to X$ as a F^{θ}-equivariant bundle on Y, and also there is a bijective correspondence between sections of $E(G/G^{\theta}) \to X$ and F^{θ}-equivariant sections of $E(G/G^{\theta}) \to Y$. We refer the reader to [BGPGMiR23] for more details on these correspondences. We have proved the following.

Proposition 2.2.9. *Any multiplicative* (G, θ)*-Higgs bundle is induced from a multiplicative* $(G, \theta)_0$*-Higgs bundle on a certain* F^{θ}*-Galois cover* $Y \to X$.

The moduli space

A moduli space of multiplicative (G, θ)-Higgs bundles can be realized inside the moduli space of G-Higgs bundles by extending the structure group. More precisely, there is a natural map

$$\begin{aligned} \mathscr{M}_{\mathbf{d}}(G, \theta) &\longrightarrow \mathscr{M}_{\mathbf{d}}(G) \\ (\mathbf{D}, E, \varphi) &\longmapsto (\mathbf{D}, E, \varphi) \end{aligned}$$

sending a multiplicative (G, θ)-Higgs bundle (E, φ) to the multiplicative G-Higgs bundle (E_G, φ_G), where

$$E_G = E \times_{G^{\theta}} G$$

is the natural extension of the structure group of E from G^{θ} to G and, regarding φ as a G^{θ}-equivariant map $f : E|_{X \setminus |D|} \to M^{\theta}$, the section φ_G is determined by the map

$$\begin{aligned} f_G : E_G|_{X \setminus |D|} &\longrightarrow G \\ [e, g] &\longmapsto g f(e) g^{-1}. \end{aligned}$$

Here, as in Section 1.2, we are identifying the symmetric variety G/G^{θ} with the subvariety $M^{\theta} = \tau^{\theta}(G)$ so that for any $g \in G^{\theta}$, the image $gf(e)g^{-1}$ stays in M^{θ}.

Recall that for any involution $\theta \in \mathrm{Aut}_2(G)$ there is a natural inclusion of cocharacters $X_*(A_{G^\theta}) \subset X_*(T)$, for $A \subset G$ a maximal θ-split torus and T a maximal torus containing it. Now, the map $\mathscr{M}_{\mathbf{d}}(G,\theta) \to \mathscr{M}_{\mathbf{d}}(G)$ restricts to a map

$$\mathscr{M}_{\mathbf{d},\lambda}(G,\theta) \longrightarrow \mathscr{M}_{\mathbf{d},w_0\lambda}(G).$$

Indeed, we have the following lemma.

Lemma 2.2.10. *For any multiplicative* (G,θ)*-Higgs bundle* (E,φ) *and any singular point* x *of it, we have* $\mathrm{inv}_x(\varphi_G) = w_0\,\mathrm{inv}_x(\varphi)$.

Proof. The decomposition $(G/G^\theta)(F) = \bigsqcup_{\lambda \in X_*(A_{G^\theta})_-} G(\mathcal{O})z^\lambda$ can be reformulated in terms of M^θ as

$$M^\theta(F) = \bigsqcup_{\lambda \in X_*(A_{G^\theta})_-} G(\mathcal{O}) *_\theta z^\lambda.$$

Now, the inclusion $M^\theta \subset G$ induces an inclusion $M^\theta(F) \subset G(F)$ and it is clear that each orbit $G(\mathcal{O}) *_\theta z^\lambda$ is mapped inside the orbit $G(\mathcal{O})z^\lambda G(\mathcal{O}) = G(\mathcal{O})z^{w_0\lambda}G(\mathcal{O})$. □

We can then define a moduli space $\mathbf{M}_{\mathbf{D},\lambda}(G,\theta)$ for multiplicative (G,θ)-Higgs bundles with singularities in $\mathbf{D}$ as the intersection of the moduli space of simple pairs $\mathbf{M}_{\mathbf{D},\lambda}(G)$ with the image of the map $\mathbf{M}_{\mathbf{d},\lambda}(G,\theta)_{\mathbf{D}} \to \mathbf{M}_{\mathbf{d},\lambda}(G)_{\mathbf{D}}$.

2.3 INVOLUTIONS AND FIXED POINTS, I

Involutions on the moduli space

Let G be a reductive algebraic group over $\mathbb{C}$ and $\theta \in \mathrm{Aut}_2(G)$ an involution. Let X be a smooth complex projective curve. To any multiplicative G-Higgs bundle (E,φ) over X we can associate other two multiplicative G-Higgs bundles obtained from (E,φ) and the action of the involution θ. These are the pairs

$$\iota_\pm^\theta(E,\varphi) = (\theta(E), \theta(\varphi)^{\pm 1}).$$

Here, $\theta(E)$ stands for the associated principal bundle $E \times_\theta G$ or, equivalently, it has the same total space as E but it is equipped with the G-action $e \cdot_\theta g = e \cdot \theta(g)$. The section $\theta(\varphi)$ is obtained as the composition of φ with the natural map

$$\begin{aligned} E(G) &\longrightarrow \theta(E)(G) \\ [e,g] &\longmapsto [e,\theta(g)]. \end{aligned}$$

Equivalently, if we regard φ as an automorphism $\varphi : E|_{X'} \to E|_{X'}$, then $\theta(\varphi)$ is set theoretically the same map as φ, but now regarded as an automorphism of $\theta(E)|_{X'}$. We can give one last interpretation of $\theta(\varphi)$ by noting that the associated G-equivariant maps $f_\varphi, f_{\theta(\varphi)} : E|_{X'} \to G$ are related by

$$f_{\theta(\varphi)} = \theta \circ f_\varphi.$$

Let $T \subset G$ be a θ-stable maximal torus and $B \subset G$ a Borel subgroup contained in a minimal θ-split parabolic and containing T. Given any dominant cocharacter

$\lambda \in X_*(T)_+$, the associated cocharacter $\theta(\lambda)$ is not dominant in general, but there is a dominant cocharacter in its orbit under the action of the Weyl group W_T. We denote this element by $\theta(\lambda)_+$. The map $\lambda \mapsto \theta(\lambda)_+$ defines an involution on $X_*(T)_+$. One checks immediately that, if x is a singular point of (E, φ), then

$$\mathrm{inv}_x(\theta(E), \theta(\varphi)^{\pm}) = (\pm\theta(\mathrm{inv}_x(E, \varphi)))_+.$$

Therefore, if $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ is a tuple of dominant coweights $\lambda_i \in X_*(T)$ with $\theta(\lambda_i)_+ = \pm\lambda_i$, then for any $\mathbf{D} \in X_{\mathbf{d}}$, we have the following involutions on the moduli space of simple pairs

$$\begin{aligned} \iota_{\pm}^{\theta} : \mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G) &\longrightarrow \mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G) \\ [E, \varphi] &\longmapsto [\theta(E), \theta(\varphi)^{\pm 1}], \end{aligned}$$

where ι_+^{θ} is well defined if $\theta(\lambda_i)_+ = \lambda_i$ and ι_-^{θ} if $(-\theta(\lambda_i))_+ = \lambda_i$. We will denote these two involutions together by writing $\iota_{\epsilon}^{\theta}$, for $\epsilon = \pm 1$.

Moreover, note that the actions of the inner automorphisms on the moduli space is trivial since, if $\alpha = \mathrm{Int}_g$ is an inner automorphism of G, then the map $e \mapsto e \cdot g^{-1}$ gives an isomorphism between any principal G-bundle E and $E \times_{\alpha} G$, commuting with any φ. Therefore, the above involutions are well defined at the level of the outer class a of θ in $\mathrm{Out}_2(G)$. That is, given any element $a \in \mathrm{Out}_2(G)$, we can define

$$\begin{aligned} \iota_{\epsilon}^{a} : \mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G) &\longrightarrow \mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G) \\ [E, \varphi] &\longmapsto [\theta(E), \theta(\varphi)^{\epsilon}], \end{aligned}$$

for any representative θ of the class a.

Remark 2.3.1. Note that it follows from the Cartan decomposition of the formal loop group $G(F)$ that if θ and θ' are two involutions representing the same element in $\mathrm{Out}_2(G)$, then $\theta(\lambda)_+ = \theta'(\lambda)_+$.

We are interested in studying the spaces $\mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G)^{\iota_{\epsilon}^{a}}$ of fixed points under these involutions. For this it will be useful for us to think about an isomorphism between the G-bundles E and $\theta(E)$, for $\theta \in \mathrm{Aut}_2(G)$, as a θ-*twisted automorphism*. By this we mean an automorphism $\psi : E \to E$ of the total space E, such that

$$\psi(e \cdot g) = \psi(e) \cdot \theta(g).$$

Therefore, a multiplicative G-Higgs bundle (E, φ) will be a fixed point of the involution ι_{ϵ}^{a} if and only if there exists a θ-twisted automorphism $\psi : E \to E$, for any element θ of the class $a \in \mathrm{Out}_2(G)$, such that the following diagram commutes

$$\begin{array}{ccc} E|_{X'} & \xrightarrow{\psi} & E|_{X'} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi^{\epsilon}} \\ E|_{X'} & \xrightarrow{\psi} & E|_{X'}. \end{array}$$

Equivalently, if we define $f_{\psi} : E \to G$ as $\psi(e) = e \cdot f_{\psi}(e)$, then $(E, \varphi) \in \mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G)^{\iota_{\epsilon}^{a}}$ if and only if, for any $e \in E$, we have

$$f_{\psi}(e)\theta(f_{\varphi}(e))f_{\psi}(e)^{-1} = f_{\varphi}(e)^{\epsilon}.$$

Fixed dominant cocharacters

Before describing the fixed points of the involution ι_-^θ, it is important that we give a good description of the involution at the level of dominant cocharacters

$$\begin{aligned} X_*(T)_+ &\longrightarrow X_*(T)_+ \\ \lambda &\longmapsto (-\theta(\lambda))_+, \end{aligned}$$

for T a θ-stable maximal torus.

As we mentioned above, in general $-\theta(\lambda)$ is not a dominant cocharacter, and that is why we need to take $(-\theta(\lambda))_+$ the dominant cocharacter in its Weyl group orbit. However, there is an exception to this, which is when there are no imaginary roots.

Indeed, the dominant Weyl chamber in $X_*(T)_+ \otimes_{\mathbb{Z}} \mathbb{R}$ is spanned by a basis of fundamental coweights $\{\mu_1, \dots, \mu_r\}$, which is the dual basis of the set of simple roots $\Delta_{\mathfrak{g}} = \{\alpha_1, \dots, \alpha_r\}$. Since θ is an involution, we have $\langle \chi^\theta, \theta(\lambda)\rangle = \langle \chi, \lambda\rangle$ for any $\chi \in X^*(T)$ and any $\lambda \in X_*(T)$. Therefore, $\{\theta(\mu_1), \dots, \theta(\mu_r)\}$ is the dual basis of $\left\{\alpha_1^\theta, \dots, \alpha_r^\theta\right\}$. Recall now that there is an involution σ on the set of simple roots which are not imaginary such that $\alpha_i^\theta + \alpha_{\sigma(i)}$ is an imaginary root. Thus, if there are no imaginary roots, we conclude that $\alpha_i^\theta = -\alpha_{\sigma(i)}$ and $\theta(\mu_i) = -\mu_{\sigma(i)}$. Summing up, we have the following.

Lemma 2.3.2. *If $\Phi_{\mathfrak{g}}^\theta = \varnothing$, then for any dominant cocharacter $\lambda \in X_*(T)_+$, the cocharacter $-\theta(\lambda)$ is also dominant, and thus $(-\theta(\lambda))_+ = -\theta(\lambda)$.*

Recall from Proposition 1.3.7 that for any involution $\theta \in \mathrm{Aut}_2(G)$ there exists an inner automorphism $u_\theta \in \mathrm{Int}(G)$ such that the involution $\theta_q := u_\theta \circ \theta$ is quasisplit. Therefore, if we let T_q be a maximal θ_q-stable torus, the involution $\lambda \mapsto (-\theta_q(\lambda))_+$ is just the involution $\lambda \mapsto -\theta_q(\lambda)$. The fixed points of this involution are simply the dominant cocharacters $X_*(A_q)_+$, where A_q is the maximal θ-split torus formed by the anti-fixed points of θ_q in T_q.

The fixed points of the involution $\lambda \mapsto -\theta(\lambda)$ in $X_*(T)$ are the cocharacters $X_*(A)$ of the maximal θ-split torus A of anti-fixed points of θ in T. Any two maximal tori are conjugate and thus there is a canonical bijection $X_*(T)_+ \to X_*(T_q)_+$. It follows from the above that under this bijection the dominant cocharacters $X_*(A)_+$ are mapped inside $X_*(A_q)_+$, but in general not every element of $X_*(A_q)_+$ comes from an element of $X_*(A)_+$. In fact, the image of $X_*(A)_+$ under this map is the set

$$X_*(A_q)_+^{u_\theta} = \left\{\lambda \in X_*(A_q)_+ : u_\theta(\lambda) = \lambda\right\}.$$

Given an element $a \in \mathrm{Out}_2(G)$, a quasisplit representative θ_q of a and a dominant cocharacter $\lambda \in X_*(A_q)_+$, we can consider the set

$$a_\lambda = \left\{\theta \text{ representative of } a : u_\theta(\lambda) = \lambda\right\}.$$

That is, the elements of a_λ are the involutions θ for which λ is in the image of the cocharacters of a maximal θ-split torus. This set clearly descends to the clique $\mathrm{cl}^{-1}(a)$ and we can define

$$\mathrm{cl}^{-1}(a)_\lambda = \left\{[\theta] \in \mathrm{cl}^{-1}(a) : u_\theta(\lambda) = \lambda\right\}.$$

The fixed points

We move on now to describe the fixed points of the involutions ι_ϵ^a, for $a \in \mathrm{Out}_2(G)$. A first result is clear.

Proposition 2.3.3. *Let $\theta \in \mathrm{Aut}_2(G)$ be a representative of a, $T \subset G$ a maximal θ-stable torus and $B \subset G$ a Borel subgroup contained in a minimal θ-split parabolic and containing T. Let $\boldsymbol{\lambda}$ be a tuple of dominant cocharacters of T.*

1. *If the $\lambda_i \in X_*(T^\theta)$, the subspace $\widetilde{\mathbf{M}}_{\mathbf{D},\boldsymbol{\lambda}}(G^\theta) \subset \mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G)$, defined as the intersection of $\mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G)$ with the image of the moduli stack of multiplicative G^θ-bundles of type $(\mathbf{D}, \boldsymbol{\lambda})$ under extension of the structure group, is contained in the fixed point subspace $\mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G)^{\iota_+^a}$.*
2. *Any extension (E_G, φ_G) of a pair (E, φ) where E is a G^θ-bundle and φ is a section of $E|_{X\setminus|D|}(S^\theta)$, for the conjugation action of G^θ on S^θ, is a fixed point of ι_-^a. In particular, if $\boldsymbol{\lambda} = (-\theta(\boldsymbol{\lambda}))_+$, $\mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G, \theta) \subset \mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G)^{\iota_-^a}$.*

Proof. Notice first that if $E_G = E \times_{G^\theta} G$ is the extension of a G^θ bundle E, then the map $\psi : E_G \to E_G$ defined as $\psi(e) = e$ for $e \in E$ and $\psi(e \cdot g) = e \cdot \theta(g)$ is clearly a well-defined θ-twisted automorphism of E_G. Now, if for every $e \in E$ we have $\theta(f_\varphi(e)) = f_\varphi(e)^\epsilon$, then

$$\begin{aligned} f_{\varphi_G}(e \cdot g)^\epsilon &= g^{-1} f_\varphi(e)^\epsilon g = g^{-1}\theta(f_\varphi(e))g \\ &= g^{-1}\theta(g)\theta(f_\varphi(e \cdot g))\theta(g)^{-1}g \\ &= f_\psi(e \cdot g)\theta(f_\varphi(e \cdot g))f_\psi(e \cdot g)^{-1}. \end{aligned}$$

And thus (E_G, φ_G) is a fixed point of ι_ϵ^a. □

The main step towards the description of the fixed points is provided by the following theorem.

Theorem 2.3.4. *If (E, φ) is a simple multiplicative G-Higgs bundle with $(E, \varphi) \cong \iota_\epsilon^a(E, \varphi)$, then:*

1. *There exists a unique $[\theta] \in \mathrm{cl}^{-1}(a)$ such that there is a reduction of structure group of E to a G^θ-bundle $E_\theta \subset E$.*
2. *If we consider the corresponding G-equivariant map $f_\varphi : E|_{X'} \to G$, then $f_\varphi|_{E^\theta}$ takes values into G^θ if $\epsilon = 1$, and S^θ if $\epsilon = -1$.*

More precisely, when $\epsilon = -1$, $f_\varphi|_{E_\theta}$ takes values in a single orbit $M_s^\theta \subset S^\theta$, for some $s \in S^\theta$ unique up to θ-twisted conjugation.

Remark 2.3.5. The statement (1) of the above theorem regarding the reduction of the structure group of E was proven in [GPR19, Proposition 3.9].

Proof. Let $\theta_0 \in \mathrm{Aut}_2(G)$ be any representative of the class a. By hypothesis, there exists some θ_0-twisted automorphism $\psi : E \to E$ making the following diagram commute

$$\begin{array}{ccc} E|_{X'} & \xrightarrow{\psi} & E|_{X'} \\ \downarrow{\varphi} & & \downarrow{\varphi^{\epsilon}} \\ E|_{X'} & \xrightarrow{\psi} & E|_{X'}. \end{array}$$

Here, $X' = X \setminus |D|$ denotes the complement of the singularity divisor of φ. Now, independently of the value of ϵ, we have that the following diagram commutes

$$\begin{array}{ccc} E|_{X'} & \xrightarrow{\psi^2} & E|_{X'} \\ \downarrow{\varphi} & & \downarrow{\varphi} \\ E|_{X'} & \xrightarrow{\psi^2} & E|_{X'}, \end{array}$$

and $\psi^2(e\cdot g) = \psi(\psi(e)\cdot\theta_0(g)) = \psi^2(e)\cdot g$, so ψ^2 is an automorphism of (E, φ). Since by assumption (E, φ) is simple, there exists some $z \in Z_G$ such that $\psi^2(e) = e \cdot z$ for every $e \in E$. Therefore,

$$e \cdot z = \psi^2(e) = \psi(e \cdot f_\psi(e)) = \psi(e) \cdot \theta_0(f_\psi(e)) = e \cdot [f_\psi(e)\theta_0(f_\psi(e))].$$

We conclude that f_ψ maps E into S_{θ_0}.

Note that f_ψ is G-equivariant for the θ_0-twisted conjugation action. Therefore, it descends to a morphism $X = E/G \to S_{\theta_0}/(G \times Z_G) \cong \mathrm{cl}^{-1}(a)$. Now, since $\mathrm{cl}^{-1}(a)$ is finite and X is irreducible, this maps X to a single element $[\theta] \cong \mathrm{cl}^{-1}(a)$. Let us take $r \in S_{\theta_0}$ a representative of the class corresponding to $[\theta]$. In other words, we take r such that $\theta = \mathrm{Int}_r \circ\theta_0$.

Let us consider the subset $E_\theta = f_\psi^{-1}(r) \subset E$. If $e \in E_\theta$, then another element $e \cdot g$ in the same fibre of E belongs to E_θ if and only if

$$r = f_\psi(e \cdot g) = g^{-1} *_{\theta_0} f_\psi(e) = g^{-1} r\theta_0(g).$$

Thus, $e \cdot g \in E_\theta$ if and only if $g = \mathrm{Int}_r \circ\theta_0(g) = \theta(g)$. That is, if $g \in G^\theta$. Therefore, E_θ defines a principal G^θ-bundle to which E reduces. This proves (1).

As we explained above, if ψ determines the isomorphism between (E, φ) and $(\theta_0(E), \theta_0(\varphi)^\epsilon)$, then for every $e \in E$ we must have

$$f_\psi(e)\theta_0(f_\varphi(e))f_\psi(e)^{-1} = f_\varphi(e)^\epsilon.$$

Thus, if $e \in E_\theta = f_\psi^{-1}(r)$, we have

$$\theta(f_\varphi(e)) = r\theta_0(f_\varphi(e))r^{-1} = f_\psi(e)\theta_0(f_\varphi(e))f_\psi(e)^{-1} = f_\varphi(e)^\epsilon,$$

proving (2).

Suppose now that $\epsilon = -1$. In that case we have a map $f_\varphi|_{E_\theta} : E_\theta|_{X'} \to S^\theta$. Since this map is G^θ-equivariant, we can quotient by G^θ and obtain a map from $X' = E_\theta|_{X'}/G^\theta$ to the quotient of S^θ by the conjugation action of G^θ. We can further quotient S^θ by the θ-twisted conjugation action and obtain a well defined map $X' \to S^\theta/G$. Again, since X' is irreducible and S^θ/G is finite, we conclude that f_φ maps $E_\theta|_{X'}$ to a single θ-twisted orbit M_s^θ, for some element $s \in S^\theta$. □

Remark 2.3.6. Notice that if there exists some $e \in E$ such that $f_\varphi(e)$ is semisimple or unipotent, then θ can be chosen so that $f_\varphi|_{E_\theta}$ takes values in the symmetric variety M^θ. Indeed, in that case we can choose r to be $r = f_\psi(e)$, so that $e \in E_\theta$, for $\theta = \mathrm{Int}_r \circ \theta_0$. We know from the above that $s = f_\varphi(e) \in S^\theta$ and that f_φ maps $E_\theta|_{X'}$ to the θ-twisted orbit M^θ_s. However, if s is semisimple or unipotent, we have by Richardson [Ric82b, Lemmas 6.1-6.3] that in fact $s \in M^\theta$ and thus $M^\theta_s = M^\theta$.

When θ and $\theta' = \mathrm{Int}_g \circ \theta \circ \mathrm{Int}_g^{-1}$ are two involutions of G related by the equivalence relation $\sim$, we have $\widetilde{\mathbf{M}}_{\mathbf{D},\boldsymbol{\lambda}}(G^\theta) = \widetilde{\mathbf{M}}_{\mathbf{D},\boldsymbol{\lambda}}(G^{\theta'})$ and $\mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G,\theta) = \mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G,\theta')$. Therefore, the above theorem allows us to decompose $\mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G)^{\iota_a^+}$ in the components $\widetilde{\mathbf{M}}_{\mathbf{D},\boldsymbol{\lambda}}(G^\theta)$, for θ the elements of the clique $\mathrm{cl}^{-1}(a)$. That is, we get

$$\mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G)^{\iota_a^+} = \bigcup_{[\theta]\in \mathrm{cl}^{-1}(a)} \widetilde{\mathbf{M}}_{\mathbf{D},\boldsymbol{\lambda}}(G^\theta).$$

The situation for $\epsilon = -1$ is a little bit more involved, as we explain now.

Definition 2.3.7. Let (G,θ,s) be a triple consisting of a reductive group G, an involution $\theta \in \mathrm{Aut}_2(G)$ and an element $s \in S^\theta$. A *multiplicative* (G,θ,s)*-Higgs pair* (E,φ) on X is a pair consisting of a principal G^θ-bundle $E \to X$, and a section φ of the bundle $E(M^\theta_s)$ associated to the conjugation action of G on M^θ_s, and defined over the complement X' of a finite subset of X.

Remark 2.3.8. Note that this is a new kind of object, different from the others defined in this paper. Indeed these are not multiplicative (G,θ)-Higgs bundles since the bundle E is a G^θ-bundle but the variety M^θ_s, although it is a symmetric variety, is equal to $M^\theta_s = M^{\theta_s}s \cong G/G^{\theta_s}$, so its stabilizer is not equal to G^θ.

To a multiplicative (G,θ,s)-Higgs pair we can associate an invariant $\mathrm{inv}(\varphi)$, which is a divisor with values in the quotient $M^\theta_s(F)/G(\mathcal{O})$. Now, since $M^\theta_s = M^{\theta_s}s$, this quotient can in fact be identified with the semigroup $X_*(A_{G^{\theta_s}})_-$, and the invariant $\mathrm{inv}(\varphi)$ is a $X_*(A_{G^{\theta_s}})_-$-valued divisor. Here, A is a maximal θ_s-split torus.

We denote by $\mathscr{M}_{\mathbf{d},\boldsymbol{\lambda}}(G,\theta,s)_{\mathbf{D}}$ the moduli stack of multiplicative (G,θ,s)-Higgs pairs with singularity type $(\mathbf{D},\boldsymbol{\lambda})$, for $\mathbf{D} \in X_{\mathbf{d}}$, where $\boldsymbol{\lambda}$ is a tuple of anti-dominant cocharacters in $X_*(A_{G^{\theta_s}})_-$. As in the case of multiplicative (G,θ)-Higgs bundles, there is a natural map

$$\mathscr{M}_{\mathbf{d},\boldsymbol{\lambda}}(G,\theta,s)_{\mathbf{D}} \longrightarrow \mathscr{M}_{\mathbf{d},w_0\boldsymbol{\lambda}}(G)_{\mathbf{D}}.$$

We denote by $\mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G,\theta,s) \subset \mathbf{M}_{\mathbf{D},w_0\boldsymbol{\lambda}}(G)$ the intersection of the image of this map with $\mathbf{M}_{\mathbf{D},w_0\boldsymbol{\lambda}}(G)$.

In this language, we can write the consequences of the results of this section as follows.

Corollary 2.3.9. *Let θ_q be the quasisplit involution representing the class $a \in \mathrm{Out}_2(G)$ and let $\boldsymbol{\lambda}$ be a tuple of anti-dominant cocharacters $\lambda_i \in X_*(A_{G^{\theta_q}})_+$, for A a maximal θ_q-split torus. Then,*

$$\mathbf{M}_{\mathbf{D},w_0\boldsymbol{\lambda}}(G)^{\iota_a^-} = \bigcup_{[\theta]\in \mathrm{cl}^{-1}(a)} \bigcup_{[s]\in (S^\theta/G)_{\boldsymbol{\lambda}}} \mathbf{M}_{\mathbf{D},\boldsymbol{\lambda}}(G,\theta,s).$$

Here, $(S^\theta/G)_\lambda$ *denotes the set*

$$(S^\theta/G)_\lambda = \left\{[s] \in S^\theta/G : u_{\theta_s}(\lambda) = \lambda\right\},$$

where $\theta_s = \mathrm{Int}_s \circ\theta$ *and* u_{θ_s} *is the inner automorphism such that* $\theta_q = u_{\theta_s} \circ \theta_s$.

Proof. Indeed, note that λ can only be a valid invariant for a multiplicative (G, θ, s)-Higgs pair if and only if the λ_i are in the image of $X_*(A^s_{G^{\theta_s}})$, for A^s a maximal θ_s-split torus, thus, if and only if $\theta_s \in a_{\lambda_i}$ for every $i = 1, \ldots, n$. □

The Hurtubise–Markman symplectic structure and involutions

We recall from the description of the deformation theory of simple multiplicative Higgs bundles that we explained in Section 2.1 that the tangent space to the moduli space $\mathbf{M}_{D,\lambda}(G)$ at a point $[E, \varphi]$ is isomorphic to the first hypercohomology $\mathbb{H}^1(C_{[E,\varphi]})$ of a certaing deformation complex. Moreover, by taking an acyclic analytic cover of X, that we denote by $\{U_i\}_i$, we have a precise description of this hypercohomology space as parametrizing some equivalence classes of pairs $(\mathbf{s}, \mathbf{t})$, with $\mathbf{s} = (s_{ij})_{i,j}$, $\mathbf{t} = (t_i)_i$, with the $s_{ij} \in \Gamma(U_i \cap U_j, E(\mathfrak{g}))$ and $t_i \in \Gamma(U_i, \mathrm{ad}(E, \varphi))$, satisfying some relations. The corresponding deformation over $X \times \mathrm{Spec}(\mathbb{C}[\delta])$ is determined as

$$\begin{cases} \mathbf{g}^{\mathbf{s}} = \mathbf{g}(1 + \delta\mathbf{s}), \\ \boldsymbol{\phi}^{\mathbf{t}} = \boldsymbol{\phi}(1 + \delta\mathbf{t}), \end{cases}$$

where $\mathbf{g}$ is determined by the transition functions of E and $\boldsymbol{\phi}$ by the restrictions of φ.

This explicit description of the tangent space allows us to compute the differential of the involution ι^a_ϵ. Indeed, $(\iota^a_\epsilon)_*(\mathbf{s}, \mathbf{t})$ is such that

$$\iota^a_\epsilon(\mathbf{g}^{\mathbf{s}}, \boldsymbol{\phi}^{\mathbf{t}}) = (\theta(\mathbf{g})^{(\iota^a_\epsilon)_*(\mathbf{s})}, (\theta(\boldsymbol{\phi})^\epsilon)^{(\iota^a_\epsilon)_*(\mathbf{t})}).$$

Now,

$$\iota^a_\epsilon(\mathbf{g}^{\mathbf{s}}, \boldsymbol{\phi}^{\mathbf{t}}) = (\theta(\mathbf{g})(1 + \delta\theta(\mathbf{s})), \theta(\boldsymbol{\phi})^\epsilon(1 + \delta\theta(\mathbf{t}))^\epsilon) = (\theta(\mathbf{g})^{\theta(\mathbf{s})}, (\theta(\boldsymbol{\phi})^\epsilon)^{\epsilon\theta(\mathbf{t})}),$$

where, for the last equality, we have used that, in $\mathbb{C}[\delta]$,

$$(1 + a\delta)^\epsilon = 1 + \epsilon a\delta.$$

We conclude that

$$(\iota^a_\epsilon)_*(\mathbf{s}, \mathbf{t}) = (\theta(\mathbf{s}), \epsilon\theta(\mathbf{t})).$$

From here we can observe the behaviour of the involutions ι^a_ϵ with respect to the Hurtubise-Markman symplectic structure, when we assume that X is an elliptic curve.

Theorem 2.3.10. *Suppose that* X *has genus* 1 *and let* Ω *denote the symplectic structure on* $\mathbf{M}_{D,\lambda}(G)$*. Then,*

$$(\iota^a_\epsilon)^*\Omega = \epsilon\Omega.$$

Therefore, $\mathbf{M}_{D,\lambda}(G)^{\iota^a_+}$ *is an algebraic symplectic submanifold, while* $\mathbf{M}_{D,\lambda}(G)^{\iota^a_-}$ *is an algebraic Lagrangian submanifold of* $\mathbf{M}_{D,\lambda}(G)$*.*

Proof. Recall our description of the symplectic form

$$\Omega((\mathbf{s},\mathbf{t}),(\mathbf{s}',\mathbf{t}')) = \langle s, \Psi(\mathbf{t}')\rangle - \langle \Psi(\mathbf{s}'), \mathbf{t}\rangle,$$

where Ψ is a certain isomorphism between the deformation complex and its dual that we defined in Section 2.1. Then, we have

$$\begin{aligned}
(\iota_{a}^{\epsilon})^{*}\Omega((\mathbf{s},\mathbf{t}),(\mathbf{s}',\mathbf{t}')) &= \Omega((\theta(\mathbf{s}),\epsilon\theta(\mathbf{t})),(\theta(\mathbf{s}'),\epsilon\theta(\mathbf{t}'))) \\
&= \langle \theta(\mathbf{s}), \epsilon\Psi^{\theta}(\theta(\mathbf{t}'))\rangle - \langle \Psi^{\theta}(\theta(\mathbf{s}')), \epsilon\theta(\mathbf{t})\rangle \\
&= \langle \theta(\mathbf{s}), \epsilon\theta(\Psi(\mathbf{t}'))\rangle - \langle \theta(\Psi(\mathbf{s}')), \epsilon\theta(\mathbf{t})\rangle \\
&= \epsilon[\langle \mathbf{s}, \Psi(\mathbf{t}')\rangle - \langle \Psi(\mathbf{s}'), \mathbf{t}\rangle] \\
&= \epsilon\Omega((\mathbf{s},\mathbf{t}),(\mathbf{s}',\mathbf{t}')),
\end{aligned}$$

since the bilinear form is invariant under automorphisms of $\mathfrak{g}$. □

3 MONOPOLES AND INVOLUTIONS

3.1 MINI-HOLOMORPHIC BUNDLES

The mini-complex structure

Consider the space $\mathbb{R} \times \mathbb{C}$ with natural coordinates (t, z), with $z = x + iy$.

Definition 3.1.1. Let $U \subset \mathbb{R} \times \mathbb{C}$ be an open subset. A function $U \to \mathbb{C}$ is *mini-holomorphic* if

$$\begin{cases} \partial_t f = 0, & \text{and} \\ \partial_{\bar{z}} f = 0. \end{cases}$$

An orientation-preserving diffeomorphism $F : U \to U$ is *mini-holomorphic* if for every mini-holomorphic function $f : U \to \mathbb{C}$, the composition $f \circ F$ is mini-holomorphic.

The natural projection $(t, z) \mapsto z$ is naturally a mini-holomorphic function from $\mathbb{R} \times \mathbb{C}$ to $\mathbb{C}$. More generally, mini-holomorphic functions $f(t, z) = (t'(t, z), z'(t, z))$ from $\mathbb{R} \times \mathbb{C}$ to $\mathbb{C}$ are characterized by the conditions $\partial_t t' > 0$, $\partial_t z' = 0$ and $\partial_{\bar{z}} z' = 0$.

Let now X be a Riemann surface and S^1 the unit circle, and put $Y = S^1 \times X$. The holomorphic structure on X and a choice of orientation on S^1 determine a *mini-complex structure* on Y. By this we mean that if we fix a local coordinate t on S^1 and a holomorphic coordinate z on X, and different coordinates t' on S^1 and z' on X with $\partial_t t' > 0$, then the change of coordinates $(t, z) \mapsto (t', z')$ is mini-holomorphic.

There are natural decompositions in the tangent bundle and in the space of 1-forms

$$TY = \underline{\mathbb{R}}_Y \partial_t \oplus \mathrm{pr}_X^* TX \quad \text{and} \quad \Omega^1(Y) = C^\infty(Y) dt \oplus \mathrm{pr}_X^* \Omega^1(X).$$

Moreover, the complex structure on X induces decompositions into $(1, 0)$ and $(0, 1)$ parts $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$ and $\Omega^1(X, \mathbb{C}) = \Omega^{1,0}(X) \oplus \Omega^{0,1}(X)$, so we can write

$$TY \otimes_{\mathbb{R}} \mathbb{C} = \underline{\mathbb{C}}_Y \partial_t \oplus \mathrm{pr}_X^* T^{1,0}X \oplus \mathrm{pr}_X^* T^{0,1}X$$

and

$$\Omega^1(Y, \mathbb{C}) = C^\infty(Y, \mathbb{C}) dt \oplus \mathrm{pr}_X^* \Omega^{1,0}(X) \oplus \mathrm{pr}_X^* \Omega^{0,1}(X).$$

Let us put

$$T^{1,0}Y = \underline{\mathbb{C}}_Y \partial_t \oplus \mathrm{pr}_X^* T^{1,0}X \quad \text{and} \quad \Omega^{1,0}(Y) = C^\infty(Y, \mathbb{C})dt \oplus \mathrm{pr}_X^* \Omega^{1,0}(X),$$

and

$$T^{0,1}Y = \underline{\mathbb{C}}_Y \partial_t \oplus \mathrm{pr}_X^* T^{0,1}X \quad \text{and} \quad \Omega^{0,1}(Y) = C^\infty(Y, \mathbb{C})dt \oplus \mathrm{pr}_X^* \Omega^{0,1}(X).$$

We can also extend this definition to define the exterior powers $\Omega^{i,0}(Y)$ and $\Omega^{0,i}(Y)$. This allows us to define the operators

$$\begin{aligned} \partial_Y : C^\infty(Y, \mathbb{C}) &\longrightarrow \Omega^{0,1}(Y) \\ f &\longmapsto \partial_t f + \partial_X f, \end{aligned}$$

and

$$\begin{aligned} \bar{\partial}_Y : C^\infty(Y, \mathbb{C}) &\longrightarrow \Omega^{0,1}(Y) \\ f &\longmapsto \partial_t f + \bar{\partial}_X f, \end{aligned}$$

where ∂_X and $\bar{\partial}_X$ are the ∂ and $\bar{\partial}$ operators on X. These operator can also be extended naturally to the higher exterior powers $\partial_Y : \Omega^{i,0}(Y) \to \Omega^{i+1,0}(Y)$ and $\bar{\partial}_Y : \Omega^{0,i}(Y) \to \Omega^{0,i+1}(Y)$. Moreover, dimensional restrictions imply that $\partial_Y^2 = \bar{\partial}_Y^2 = 0$. A function $f \in C^\infty(Y, \mathbb{C})$ is mini-holomorphic if $\bar{\partial}_Y f = 0$.

Mini-holomorphic principal bundles

Let G be a complex reductive group and let $p\colon \mathbb{E} \to Y$ be a smooth principal G-bundle. Let $V_{\mathbb{E}} = \ker dp \subset T\mathbb{E}$ denote the *vertical distribution*, which is naturally isomorphic as a bundle over $\mathbb{E}$ to the trivial $\mathfrak{g}$-bundle $\mathbb{E} \times \mathfrak{g}$. There is a natural short exact sequence of bundles over $\mathbb{E}$

$$0 \longrightarrow V_{\mathbb{E}} \hookrightarrow T\mathbb{E} \xrightarrow{dp} p^*TY \longrightarrow 0.$$

We recall that, by definition, a G-connection D on $\mathbb{E}$ is a splitting of the above exact sequence of bundles over $\mathbb{E}$ which is G-invariant. That is, the G-connection D determines a *horitonzal lift* $H_D : p^*TY \to T\mathbb{E}$ whose image is the *horizontal distribution*, satisfying that $H_D(p^*TY) \oplus V_{\mathbb{E}} = T\mathbb{E}$. The condition of being G-invariant means that, for any $e \in \mathbb{E}$ and $g \in G$,

$$H_D(p^*TY)_{e\cdot g} = (R_g)_* H_D(p^*TY)_e,$$

where $R_g : \mathbb{E} \to \mathbb{E}$ denotes the right multiplication by g. Equivalently, the distribution $H_D(p^*TY)$ can be defined as the kernel of a projection $V_D : T\mathbb{E} \to V_{\mathbb{E}}$. Since $V_{\mathbb{E}}$ is isomorphic to $\mathbb{E} \times \mathfrak{g}$, the projection V_D can be regarded as a 1-form $\omega_D \in \Omega^1(\mathbb{E}, \mathfrak{g})$, called the *connection 1-form* of D. The horizontal distribution is then $\ker \omega_D$. This 1-form is G-equivariant in the sense that $R_g^* \omega_D = \mathrm{ad}_{g^{-1}} \circ \omega_D$; which follows from the horizontal distribution being G-invariant.

The *curvature* F_D of D measures the failure of integrability of the horizontal distribution $H_D(p^*TY)$ and is the $\mathfrak{g}$-valued 2-form defined as

$$F_D = h^*\omega_D \in \Omega^2(\mathbb{E}, \mathfrak{g}),$$

where $h : T\mathbb{E} \to H_D(p^*TY)$ is the natural projection to the horizontal distribution. Indeed, one can show that, for $\xi, \eta \in T\mathbb{E}$,

$$F_D(\xi, \eta) = -\omega_D([h(\xi), h(\eta)]).$$

Let $dp^{-1}(T^{0,1}Y) \subset T\mathbb{E} \otimes_{\mathbb{R}} \mathbb{C}$ denote the inverse image of the $(0,1)$ distribution in Y. The above exact sequence induces a short exact sequence

$$0 \longrightarrow V_{\mathbb{E}} \otimes_{\mathbb{R}} \mathbb{C} \hookrightarrow dp^{-1}(T^{0,1}Y) \xrightarrow{dp} p^*T^{0,1}Y \longrightarrow 0.$$

Definition 3.1.2. A *mini-holomorphic structure* $D^{0,1}$ on $\mathbb{E}$ is a G-invariant splitting of the above exact sequence, such that the resulting distribution $H_{D^{0,1}}(p^*T^{0,1}Y) \subset dp^{-1}(T^{0,1}Y)$ is integrable. A pair $\mathcal{E} = (\mathbb{E}, D^{0,1})$ formed by a principal G-bundle $\mathbb{E}$ endowed with a mini-holomorphic structure is called a *mini-holomorphic (principal) G-bundle*.

Equivalently, a mini-holomorphic structure is determined by a form $\omega_{D^{0,1}} : dp^{-1}(T^{0,1}Y) \to \mathfrak{g}$, whose kernel is the distribution $H_{D^{0,1}}(p^*T^{0,1}Y)$.

Clearly, if D is a connection on $\mathbb{E}$, we can restrict the horizontal lift $H_D : p^*TY \otimes \mathbb{C} \to T\mathbb{E} \otimes \mathbb{C}$ to $p^*T^{0,1}Y$ and thus obtain a horizontal lift $H_D : p^*T^{0,1}Y \to dp^{-1}(p^*T^{0,1}Y)$. Equivalently, we can define this horizontal lift by restricting the connection 1-form ω_D to $dp^{-1}(p^*T^{0,1}Y)$. The integrability condition now amounts to the vanishing of the 2-form

$$F_D^{0,2} := F_D|_{dp^{-1}(p^*T^{0,1}Y) \times dp^{-1}(p^*T^{0,1}Y)}.$$

Associated bundles and covariant derivatives

A local description of the objects just considered in terms of covariant derivatives helps to clarify the picture. Pulling back by a local section $s : U \to \mathbb{E}$, for $U \subset Y$ a sufficiently small open subset, a connection 1-form ω_D yields a 1-form $\mathcal{A} = s^*\omega_D \in \Omega^1(U, \mathfrak{g})$. We can also pull back the curvature to obtain $F_{\mathcal{A}} = s^*F_D \in \Omega^2(U, \mathfrak{g})$, which can be shown to be equal to

$$F_{\mathcal{A}} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}.$$

(Recall that the wedge product on $\mathfrak{g}$ valued forms involves taking the wedge product on the form part and the Lie bracket on the Lie algebra part). If $\rho : G \to GL(V)$ is a representation of G, then we can consider the *covariant derivative*, defined locally as

$$\begin{aligned} d_{\mathcal{A}} : \Omega^0(U, \mathbb{E} \times_\rho V) &\longrightarrow \Omega^1(U, \mathbb{E} \times_\rho V) \\ v &\longmapsto dv + \rho(\mathcal{A})v. \end{aligned}$$

This satisfies the Leibniz rule: for any $f \in C^\infty(U)$, $d_{\mathcal{A}}(f\nu) = (df)\nu + f d_{\mathcal{A}}\nu$. Moreover, it can be extended to higher exterior powers $\Omega^i(U, \mathbb{E}\times_\rho V) \to \Omega^{i+1}(U, \mathbb{E}\times_\rho V)$, and the obstruction for this extension to define a complex is precisely the curvature; indeed

$$(d_{\mathcal{A}} \circ d_{\mathcal{A}})(\nu) = \rho(F_{\mathcal{A}})\nu.$$

Now, if $D^{0,1}$ is a mini-holomorphic structure, we can locally pull back the 1-form $\omega_{D^{0,1}}$ to obtain a 1-form $\mathcal{A}^{0,1} \in \Omega^{0,1}(U, \mathfrak{g})$ and, for any representation $\rho : G \to GL(V)$ define

$$\begin{aligned} \bar\partial_{D^{0,1}} : \Omega^0(U, \mathbb{E}\times_\rho V) &\longrightarrow \Omega^{0,1}(U, \mathbb{E}\times_\rho V) \\ \nu &\longmapsto \bar\partial\,\nu + \rho(\mathcal{A}^{0,1})\nu. \end{aligned}$$

This also satisfies a Leibniz rule: for any $f \in C^\infty(U, \mathbb{C})$, $\bar\partial_{D^{0,1}}(f\nu) = (\bar\partial_Y f)\nu + f\,\bar\partial_{D^{0,1}}\nu$. Moreover, it can also be extended to higher exterior powers and the integrability condition of the distribution defined by $D^{0,1}$ implies that $\bar\partial_{D^{0,1}} \circ \bar\partial_{D^{0,1}} = 0$. We conclude that a mini-holomorphic structure $D^{0,1}$ induces in every associated bundle a mini-holomorphic structure in the sense of Mochizuki [Moc22].

Note that if $\mathcal{A} \in \Omega^1(U, \mathfrak{g})$ is the 1-form defined by a connection on $\mathbb{E}$, the $(0,1)$-part of this form $\mathcal{A}^{0,1} \in \Omega^{0,1}(U, \mathfrak{g})$ is the form defined by the mini-holomorphic structure induced by the connection, provided that the $(0,2)$-part $F^{0,2}_{\mathcal{A}} \in \Omega^{0,2}(U, \mathfrak{g})$ of $F_{\mathcal{A}}$ vanishes.

We can do this even more explicit if we decompose

$$\mathcal{A} = \mathcal{A}^{1,0}_X + \mathcal{A}^{0,1}_X + \mathcal{A}_t dt,$$

for $\mathcal{A}_X \in \mathrm{pr}_X^* \Omega^1(X, \mathfrak{g})$ and $\mathcal{A}_t \in \Omega^0(Y, \mathfrak{g})$, and

$$d_{\mathcal{A}} = \partial_{\mathcal{A},X} + \bar\partial_{\mathcal{A},X} + \partial_{\mathcal{A},t},$$

for $\partial_{\mathcal{A},X} = \partial_X + \rho(\mathcal{A}^{1,0}_X)$, $\bar\partial_{\mathcal{A},X} = \bar\partial_X + \rho(\mathcal{A}^{0,1}_X)$ and $\partial_{\mathcal{A},t} = \partial_t + \mathcal{A}_t dt$. The curvature now is equal to

$$F_{\mathcal{A}} = d_{\mathcal{A}} \circ d_{\mathcal{A}} = [\partial_{\mathcal{A},X}, \bar\partial_{\mathcal{A},X}] + [\partial_{\mathcal{A},X}, \partial_{\mathcal{A},t}] + [\bar\partial_{\mathcal{A},X}, \partial_{\mathcal{A},t}].$$

Where we note that the squares of the components of $d_{\mathcal{A}}$ vanish by the dimensional restrictions on both X and S^1. We denote the first component of the curvature as $F_{\mathcal{A},X} = [\partial_{\mathcal{A},X}, \bar\partial_{\mathcal{A},X}]$ and thus we put

$$F_{\mathcal{A}} = F_{\mathcal{A},X} + [\partial_{\mathcal{A},X}, \partial_{\mathcal{A},t}] + [\bar\partial_{\mathcal{A},X}, \partial_{\mathcal{A},t}].$$

The $(0,2)$-part of this form is $F^{0,2}_{\mathcal{A}} = [\bar\partial_{\mathcal{A},X}, \partial_{\mathcal{A},t}]$.

We conclude that if $\mathcal{E}$ is a mini-holomorphic bundle determining an operator $\bar\partial_{\mathcal{E}}$ that, as above, we decompose as $\bar\partial_{\mathcal{E}} = \bar\partial_{\mathcal{E},X} + \partial_{\mathcal{E},t}$, then the integrability condition can be put as

$$[\bar\partial_{\mathcal{E},X}, \partial_{\mathcal{E},t}] = 0.$$

The Chern pair

Let $K \subset G$ be a maximal compact subgroup of G. Let $h\colon \mathbb{E} \to G/K$ be a G-equivariant map, which gives a reduction of the structure group of $\mathbb{E}$ from G to K (we call this a K-*reduction* on $\mathbb{E}$). Indeed, just take $\mathbb{E}_h = h^{-1}(K) \subset \mathbb{E}$, which is naturally a principal K-bundle.

Consider ∇ a K-connection on $\mathbb{E}_h$, which is determined by a horizontal lift $H_\nabla : p^*TY \to T\mathbb{E}_h$. The connection ∇ can be extended naturally to a G-connection on $\mathbb{E}$, which we also denote by ∇, by putting, for $e \in \mathbb{E}_h$, and $g \in G$,

$$H_\nabla(p^*TY)_{e\cdot g} = (R_g)_*(i_* H_\nabla(p^*TY)_e),$$

where $i : \mathbb{E}_h \hookrightarrow \mathbb{E}$ denotes the inclusion. The G-connections on $\mathbb{E}$ that are obtained in this way from K-connections on $\mathbb{E}_h$ are called h-*connections*.

Let $\mathcal{E} = (\mathbb{E}, D^{0,1})$ be a mini-holomorphic bundle. Consider now a G-invariant splitting $D^{1,0}$ of the exact sequence

$$0 \longrightarrow V_{\mathbb{E}} \otimes_{\mathbb{R}} \mathbb{C} \hookrightarrow dp^{-1}(T^{1,0}Y) \xrightarrow{dp} p^*T^{1,0}Y \longrightarrow 0,$$

determined by a $\mathfrak{g}$-valued 1-form $\omega_{D^{1,0}}$ defined on $dp^{-1}(T^{0,1}Y)$. We can now reconstruct a connection ∇ on $\mathbb{E}$ by putting

$$\omega_\nabla = \omega_{D^{1,0},X} + \omega_{D^{0,1},X} + \tfrac{1}{2}(\omega_{D^{1,0},t} + \omega_{D^{0,1},t}).$$

We can also consider a G-equivariant $\Phi \in \Omega^0(\mathbb{E}, \mathfrak{g})$ defined as

$$\Phi\, dt = -\tfrac{i}{2}(\omega_{D^{1,0},t} - \omega_{D^{0,1},t}).$$

We can write

$$\omega_D = \omega_\nabla - i\Phi\, dt = \omega_{D^{1,0},X} + \omega_{D^{0,1}},$$

which is the 1-form of a connection D with $(0,1)$-part giving the mini-holomorphic structure of $\mathcal{E}$. We denote this connection by $D = \nabla - i\Phi$.

Proposition 3.1.3. *There exists a unique splitting* $D^{1,0}$ *as above, which we denote by* $D^{1,0}_h$*, such that, if* (∇, Φ) *is the pair obtained by the procedure explained above, the resulting connection* ∇ *is an* h*-connection and* Φ *is obtained by extending a* K*-equivariant section in* $\Omega^0(\mathbb{E}_h, \mathfrak{k})$*. The pair* (∇, Φ) *is called the* Chern pair *of* $\mathcal{E}$.

Moreover, if we consider the connection $D = \nabla - i\Phi$ *defined as above, then we have* $F^{2,0}_D = F^{0,2}_D = 0$.

We omit the proof of this proposition in full generality here, but remark that this is easy to see at the level of associated bundles, and thus it is true in general by Tannakian considerations. Indeed, a K-reduction on $\mathbb{E}$ induces a Hermitian metric $\langle -, - \rangle_h$ on every associated bundle $\mathbb{E} \otimes_\rho V$. Now, by an analogous of the typical argument in Hermitian geometry, there exists a unique operator

$$\partial^h_{\mathcal{E}} : \Omega^0(Y, \mathbb{E} \times_\rho V) \longrightarrow \Omega^{1,0}(Y, \mathbb{E} \times_\rho V)$$

satisfying the Leibniz rule $\partial^h_{\mathcal{E}}(fv) = (\partial_Y f)v + f\partial_{\mathcal{E}} v$ and such that

$$\bar{\partial}_Y \langle u, v\rangle_h = \langle \bar{\partial}_{\mathcal{E}}\, u, v\rangle_h + \langle u, \partial^h_{\mathcal{E}} v\rangle_h.$$

Moreover, $\partial^h_{\mathcal{E}}$ can be extended to higher exterior powers and one can easily check that $\partial^h_{\mathcal{E}} \circ \partial^h_{\mathcal{E}} = 0$. We then construct a Hermitian connection d_A on $\mathbb{E} \times_\rho V$ by considering the operator

$$d_A = \partial^h_{\mathcal{E},X} + \bar{\partial}_{\mathcal{E},X} + \tfrac{1}{2}(\partial^h_{\mathcal{E},t} + \partial_{\mathcal{E},t})$$

and a section Φ of $\mathrm{End}(\mathbb{E} \times_\rho V)$ by putting

$$\Phi = -\tfrac{i}{2}(\partial^h_{\mathcal{E},t} - \partial_{\mathcal{E},t}).$$

We have

$$\bar{\partial}_{\mathcal{E},X} = \bar{\partial}_{A,X} \quad \text{and} \quad \partial_{\mathcal{E},t} = d_{A,t} - i\Phi(-)dt.$$

And also,

$$\partial^h_{\mathcal{E},X} = \partial_{A,X} \quad \text{and} \quad \partial^h_{\mathcal{E},t} = d_{A,t} + i\Phi(-)dt.$$

At this level, the connection $D = \nabla - i\Phi$ is given by the operator

$$d_{\mathcal{A}} = d_A - i\Phi(-)dt,$$

whose $(0,1)$ part is precisely $\bar{\partial}_{\mathcal{E}}$. The integrability of $\bar{\partial}_{\mathcal{E}}$ implies that

$$F^{0,2}_{\mathcal{A}} = [\bar{\partial}_{A,X}, d_{A,t} - i\Phi(-)dt] = 0.$$

On the other hand, we can also consider the connection $\bar{D} = \nabla + i\Phi$, given by the operator $d_{\bar{\mathcal{A}}} = d_A + i\Phi(-)dt$. The $(0,1)$ part of this connection is $\partial^h_{\mathcal{E}}$ and the integrability of this operator implies that

$$F^{2,0}_{\bar{\mathcal{A}}} = [\partial_{A,X}, d_{A,t} + i\Phi(-)dt] = 0.$$

The scattering map

Let $\mathcal{E} \to Y$ be a mini-holomorphic G-bundle and $\gamma : [0,1] \to Y$ a path with $\gamma_*(\frac{\partial}{\partial s}) \subset \underline{\mathbb{R}}\partial_t$, where s is the coordinate in $[0,1]$. Consider the pull-back bundle $\gamma^*\mathcal{E} \to [0,1]$. We can also pull-back the mini-holomorphic structure of $\mathcal{E}$, which induces a connection on $\gamma^*\mathcal{E}$. Taking parallel transport of this connection induces an isomorphism

$$\varphi_{\mathcal{E},\gamma} : \mathcal{E}_{\gamma(0)} \longrightarrow \mathcal{E}_{\gamma(1)},$$

called the *scattering map* associated to $\mathcal{E}$ along γ.

In particular, for $t_1, t_2 \in [0, 2\pi]$, and for any $x \in X$, we can consider the path

$$\begin{aligned} \gamma_x : [0,1] &\longrightarrow Y = S^1 \times X \\ s &\longmapsto (e^{i[t_1 + s(t_2 - t_1)]}, x) \end{aligned}$$

and the scattering map $\varphi_{\mathcal{E},\gamma_x} : \mathcal{E}_{(e^{it_1},x)} \to \mathcal{E}_{(e^{it_2},x)}$. For any $t \in [0, 2\pi]$, we denote by $\mathcal{E}_t \to X$ the G-bundle with fibres $(\mathcal{E}_t)_x = \mathcal{E}_{(e^{it},x)}$. The distribution

$H_{\mathcal{E}}(p^*T^{0,1}Y)|_{\{e^{it}\}\times X} \subset T\mathcal{E}_t$ determines a holomorphic structure on $\mathcal{E}_t$. We denote the resulting holomorphic bundle by $E_t \to X$. Putting all the γ_x together yields an isomorphism

$$\varphi_{\mathcal{E},t_1,t_2} : E_{t_1} \longrightarrow E_{t_2}.$$

By construction, the differential of this map preserves the holomorphic structures and therefore is holomorphic. In particular, $\varphi_{\mathcal{E},0,2\pi}$ defines a holomorphic automorphism of E_0.

Dirac-type singularities

Consider now an interval a disc $\Delta_\delta \subset \mathbb{C}$ of radius δ and centered at 0, and put $U = \mathbb{R} \times \Delta_\delta$ and $U' = U \setminus \{(0,0)\}$. Let $\mathcal{E} \to U'$ be a mini-holomorphic G-bundle. For any $t \in \mathbb{R}$ we can take the corresponding holomorphic G-bundle $E_t \to U'$. There is a scattering map

$$\varphi_{\mathcal{E},-1,1} : E_{-1} \longrightarrow E_1,$$

This map is holomorphic over Δ^*_δ and has a singularity at 0.

Definition 3.1.4. We say that $\mathcal{E}$ has a *Dirac-type singularity* at 0 if the map $\varphi_{\mathcal{E},-1,1}$ is meromorphic over Δ_δ.

By taking trivializations of E_{-1} and E_1, the scattering map is determined by a meromorphic map $\Delta_\delta \to G$. Taking limit as $\delta \to 0$ we obtain an element of the formal loop group $G(F)$, well defined up to the left and right multiplication action of $G(\mathcal{O})$ determined by the choice of trivializations. Thus, if we fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, we obtain a dominant cocharacter $\lambda \in X_*(T)_+$, which we call the *weight* of the Dirac-type singularity.

Let now X be a compact Riemann surface and put $Y = S^1 \times X$. Let $n \in \mathbb{Z}_+$ and take n pairwise different points $x_1, \dots, x_n \in X$ and n real numbers $0 < t_1 \leq \cdots \leq t_n < 2\pi$. Put $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{t} = (t_1, \dots, t_n)$, $y_i = (e^{it_i}, x_i) \in Y$, $\mathbf{y} = (y_1, \dots, y_n)$ and $Y' = Y \setminus \{y_1, \dots, y_n\}$. Consider also a tuple of n dominant cocharacters $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$, with $\lambda_i \in X^*(T)_+$.

Definition 3.1.5. A *singular mini-holomorphic bundle on* Y *of type* $(\mathbf{t}, \mathbf{x}, \boldsymbol{\lambda})$ is a mini-holomorphic bundle $\mathcal{E}$ over Y' such that, for each $i = 1, \dots, n$, taking local coordinates in some neighbourhood of the point y_i, the resulting mini-holomorphic bundle over $U' = (\mathbb{R} \times \Delta_\delta) \setminus \{(0,0)\}$ has a Dirac-type singularity of weight λ_i at 0.

Mini-holomorphic bundles and multiplicative Higgs bundles

Multiplicative Higgs bundles were defined in the previous chapter in the algebraic context, as objects over a smooth complex projective curve. The notion can be immediately translated to the holomorphic context; indeed, a multiplicative G-Higgs bundle of type $(\mathbf{x}, \boldsymbol{\lambda})$ over a compact Riemann surface X is a pair (E, φ) formed by a holomorphic principal G-bundle $E \to X$ and a meromorphic automorphism of E, holomorphic over $X \setminus \{x_1, \dots, x_n\}$ and such that the invariant obtained at each x_i by restricting φ to a formal disc around it is equal to λ_i. One is easily convinced by GAGA-style arguments that both the algebraic notion and the holomorphic notion are in fact the same. To be more precise, we mean that an algebraic multiplicative Higgs bundle over a smooth complex projective curve determines a holomorphic multiplicative Higgs bundle over its analytification, and viceversa, every holomorphic multiplicative Higgs bundle over a compact Riemann surface arises in this way.

A singular mini-holomorphic G-bundle $\mathcal{E}$ on Y of type $(\mathbf{t}, \mathbf{x}, \boldsymbol{\lambda})$ determines a multiplicative G-Higgs bundle (E, φ) on X of type $(\mathbf{x}, \boldsymbol{\lambda})$ by putting $E = E_0$ and taking φ as the scattering map $\varphi = \varphi_{\mathcal{E},0,2\pi} : E_0 \to E_{2\pi} = E_0$. It is clear that we can go in the other direction by using φ to glue $\mathrm{pr}_X^* E \to [0, 2\pi] \times X$ along $\{0\} \times X'$ and $\{2\pi\} \times X'$ to obtain a mini-holomorphic bundle $\mathcal{E} \to Y'$ yielding φ as the scattering map. Summing up, we have the following.

Proposition 3.1.6. *For any $\mathbf{t} = (t_1, \ldots, t_n)$ there is an equivalence of categories between the category of singular mini-holomorphic G-bundles on $Y = S^1 \times X$ of type $(\mathbf{t}, \mathbf{x}, \boldsymbol{\lambda})$ and the category of multiplicative G-Higgs bundles on X of type $(\mathbf{x}, \boldsymbol{\lambda})$.*

3.2 MONOPOLES AND THE CHS CORRESPONDENCE

The Dirac monopole

Let $T_K \cong U(1)^r$ be a real compact r-dimensional torus and consider a real cocharacter λ of T_K. The real cocharacter λ can be thought of as an element $i\lambda \in \mathfrak{t}_K$ of the Lie algebra $\mathfrak{t}_K \cong i\mathbb{R}^r$ of T_K, defining a map $U(1) \to T_K$ of the form $e^{it} \mapsto \exp(it\lambda)$.

Consider the punctured 3-dimensional Euclidean space $\mathbb{R}^3 \setminus \{0\}$, endowed with Cartesian coordinates (x, y, t) and spherical coordinates (r, ϕ, θ) with

$$(x, y, t) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta).$$

For any $\varepsilon > 0$ let us consider the open subsets

$$U_{\pm,\varepsilon} = \left\{(x, y, t) \in \mathbb{R}^3 \setminus \{0\} : \pm t > \mp\varepsilon\right\},$$

which intersect in

$$U_{+-,\varepsilon} = \left\{(x, y, t) \in \mathbb{R}^3 \setminus \{0\} : t \in (-\varepsilon, \varepsilon)\right\}.$$

We can define now a T_K-bundle $P_\lambda \to \mathbb{R}^3 \setminus \{0\}$, determined by the transition function

$$\begin{aligned} g_\lambda : U_{+-} &\longrightarrow T_K \\ (r, \phi, \theta) &\longmapsto \exp(i\phi\lambda). \end{aligned}$$

Consider now the $\mathfrak{t}_K$-valued (for $\mathfrak{t}_K$ the Lie algebra of T_K) 1-forms $A_\pm \in \Omega^1(U_\pm, \mathfrak{t})$ defined as

$$A_\pm = \tfrac{i}{2}\lambda(\pm 1 + \cos\theta)d\phi.$$

It is easy to check that these define a T_K-connection ∇ on P_λ. We can also consider the field

$$\Phi(r, \phi, \theta) = \frac{i\lambda}{2r} \in C^\infty(\mathbb{R}^3 \setminus \{0\}, \mathfrak{t}_K).$$

Definition 3.2.1. The tuple $(P_\lambda, \nabla, \Phi)$ is called the *standard Dirac* T_K*-monopole of charge* λ on $\mathbb{R}^3 \setminus \{0\}$.

The curvature of the connection ∇ is equal to

$$F_\nabla = \tfrac{i}{2}\lambda d(\cos\theta d\phi) = -\tfrac{i}{2}\lambda \sin\theta d\theta \wedge d\phi = -\tfrac{i}{2}\lambda d\Omega,$$

where the 1-form $d\Omega = \sin\theta d\theta \wedge d\phi$ is the area form on the unit sphere. In particular, one check easily that the *Bogomolny equation* holds

$$F_\nabla = *(d\Phi),$$

where $*$ denotes the Hodge star operator taking 1-forms to 2-forms.

Consider now a real character χ of T_K, which we think of as an element $\chi \in \mathfrak{t}_K^* \cong \mathbb{R}^r$, defining a map $T_K \to U(1)$ of the form $\exp(ix) \mapsto e^{i\langle\chi,x\rangle}$. The character χ allows us to consider a 2-form $\langle\chi, F_\nabla\rangle = \frac{i}{2}\langle\chi,\lambda\rangle d\Omega \in \Omega^2(\mathbb{R}^3 \setminus \{0\})$, and integrating this on the unit sphere we obtain an invariant

$$\int_{S^2} \langle\chi, F_\nabla\rangle = -\tfrac{i}{2}\langle\chi,\lambda\rangle \int_{S^2} d\Omega = -2\pi i\langle\chi,\lambda\rangle.$$

Let $T \cong (\mathbb{C}^*)^r$ be the complex torus obtained as the complexification of T_K. The T_K-bundle P_λ admits an extension of the structure group to a principal T-bundle $\mathbb{E}_\lambda$, with a natural reduction of the structure group h such that $P_\lambda = \mathbb{E}_{\lambda,h}$ and the connection ∇ extends to an h-connection on $\mathbb{E}_\lambda$. The field Φ can be also extended to define a section of $\mathbb{E}_\lambda(\mathfrak{t})$. We can now consider the connection $D = \nabla - i\Phi$ on $\mathbb{E}_\lambda$ obtained from

$$\mathcal{A}_\pm = A_\pm - i\Phi dt = A_\pm - i\Phi \sin\theta d\theta.$$

The curvature of D is equal to

$$F_{\mathcal{A}} = F_A - id(\Phi dt) = -\frac{i}{2}\lambda \sin\theta d\theta \wedge d\phi + \frac{\lambda}{2r^2}\sin\theta dr \wedge d\theta.$$

By regarding $\mathbb{R}^3$ as $\mathbb{R} \times \mathbb{C}$ we can consider another set of coordinates (z, t), where $z = x + iy$ is the complex coordinate on $\mathbb{C}$. In these coordinates, we obtain

$$F_A = \frac{\lambda}{4r^3}(t dz \wedge d\bar{z} - \bar{z} dz \wedge dt + z d\bar{z} \wedge dt),$$

and

$$-id(\phi dt) = -\frac{\lambda}{4r^3}(\bar{z} dz \wedge dt + z d\bar{z} \wedge dt).$$

And therefore

$$F_{\mathcal{A}} = \frac{\lambda}{4r^3}(t dz \wedge d\bar{z} + 2\bar{z} dz \wedge dt).$$

We conclude that $F^{0,2}_{\mathcal{A}} = 0$ and thus $D^{0,1}$ defines a mini-holomorphic structure on $\mathbb{E}_\lambda$. We denote the resulting mini-holomorphic bundle by $\mathcal{E}_\lambda$. Consider the complexification of the real cocharacter λ, which is a cocharacter of T, that we also denote by λ. The mini-holomorphic bundle $\mathcal{E}_\lambda$ has a Dirac-type singularity of weight λ at 0. Indeed, one can just take a suitable frame in which the parallel transport is simply the transition function $g_\lambda(r, \phi, \theta) = r\exp(i\phi\lambda)$ which induces an element of $G(F)$ in the orbit indexed by λ.

The Hermitian–Bogomolny equation

We come back now to the setting of the previous section. Let X be a compact Riemann surface with Kähler area form ω_X (that we choose normalized to $\int_X \omega_X = 1$) and put $Y = S^1 \times X$. Consider tuples of pairwise different points $x_1, \dots, x_n \in X$ and real numbers $0 < t_1 \leq \cdots \leq t_n < 2\pi$, and put $\boldsymbol{x} = (x_1, \dots, x_n)$, $\boldsymbol{t} = (t_1, \dots, t_n)$, $y_i = (e^{it_i}, x_i) \in Y$, $\boldsymbol{y} = (y_1, \dots, y_n)$ and $Y' = Y \setminus \{y_1, \dots, y_n\}$.

Let G be a complex reductive group and $K \subset G$ a maximal compact subgroup. Take $T_K \subset K$ a maximal torus, and consider the complexification $T \subset G$ which is a maximal torus of G. We also consider $B \subset G$ a Borel subgroup containing T. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$, with $\lambda_i \in X^*(T)_+$, be a tuple of dominant cocharacters.

Let $\mathbb{E} \to Y'$ be a principal G-bundle and $h : \mathbb{E} \to G/K$ a K-reduction of $\mathbb{E}$, with $\mathbb{E}_h = h^{-1}(K)$ the corresponding principal K-bundle. By a h-*pair* on $\mathbb{E}$ we mean a pair (∇, Φ), where ∇ is a h-connection on $\mathbb{E}$ and Φ is a section of $\mathbb{E}(\mathfrak{g})$ obtained from the natural extension of a section of $\mathbb{E}_h(\mathfrak{k})$. In an abuse of notation, we keep the symbol ∇ to denote the associated covariant derivative.

Definition 3.2.2. A *solution to the Hermitian–Bogomolny* (HB) equation on $\mathbb{E}$ is a h-pair (∇, Φ) such that

$$F_\nabla - *\nabla\Phi = iC\omega_X,$$

for $C \in Z(\mathfrak{k})$ some central element of the Lie algebra of K.

Remark 3.2.3. When $C = 0$, the above equation is called simply the *Bogomolny equation*; the slightly generalization we consider is called the Hermitian–Bogomolny equation in an analogy with the distinction between Einstein/Yang–Mills equations and Hermitian Einstein/Yang–Mills equations. As we explained above, the standard Dirac monopole is a solution to the Bogomolny equation.

Definition 3.2.4. A h-pair (∇, Φ) on $\mathbb{E}$ has *singularities of Dirac type* $(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{\lambda})$ if

1. for each $j = 1, \dots, n$ there exists a neighbourhood B_j of y_j diffeomorphic to a 3-ball and over which $\mathbb{E}$ admits a reduction to a T-bundle isomorphic to the bundle $\mathbb{E}_{\lambda_j}$ coming from the standard singular monopole of charge λ_j, and

2. under this isomorphism, the field Φ takes the form

$$\Phi = -\frac{i\lambda_j}{2r} + O(1)$$

   and we have

$$\nabla(r\Phi) = O(1).$$

We say that the h-pair (∇, Φ) is a *singular* h-*monopole on* $\mathbb{E}$ *of type* $(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{\lambda})$ if it has singularities of Dirac type $(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{\lambda})$ and it is a solution to the HB equation.

Remark 3.2.5. The above definition ensures that a pair with singularities with Dirac type can be locally approximated by a standard Dirac monopole, in such a way that the curvature is $O(r^{-2})$, so it can be integrated around a singularity.

Topological restrictions

Singular monopoles satisfy several topological restrictions. The first one has to do with the possible values for the charges of a Dirac type singularity, which is restricted due to the compacity of the base manifold Y. First, we make some comments regarding characters.

Let W denote the Weyl group of G and consider $X_*(T)^W \subset X_*(T)$ the subgroup of characters which are W-invariant. We remark that the W-invariant characters, regarded as characters of G are G-invariant under the adjoint action of G on itself. Therefore, for any character $\chi \in X^*(T)^W = X^*(G)^G$ and for any k-form on Y α with values in the adjoint bundle $E(\mathfrak{g})$, we can define a k-form $\langle \chi, \alpha \rangle \in \Omega^k(Y)$. Moreover, there is a natural isomorphism between W-invariant characters and the dual of the centre

$$X^*(T)^W \cong Z_G^\vee = X^*(T)/X^*(T^{ad}).$$

We call these characters *central characters*.

We denote by G^{sc} the universal cover of G and by T^{sc} its maximal torus. Recall that the coroot lattice of G is equal to the cocharacter lattice $X_*(T^{sc})$, which is naturally included inside $X_*(T)$ and can be identified with the common kernel of the central characters:

$$X_*(T^{sc}) = \left\{ \lambda \in X_*(T) : \langle \chi, \lambda \rangle = 0, \forall \chi \in Z_G^\vee \right\}.$$

Indeed, the quotient of $X_*(T)$ by the common kernel of the central characters is equal to the dual of $Z_G^\vee$ under vector space duality between $\mathfrak{t}$ and $\mathfrak{t}^*$, which is equal to $\pi_1(G) = X_*(T)/X_*(T)^{sc}$.

Proposition 3.2.6. *If (∇, Φ) is an $\mathfrak{h}$-pair on $\mathbb{E}$ with singularities of Dirac type $(\mathbf{t}, \mathbf{x}, \boldsymbol{\lambda})$, then*

$$\sum_{i=1}^{n} \lambda_i \in X_*(T^{sc})_+.$$

Proof. Take a sufficiently small ball B_i around each y_i such that $\mathbb{E}$ can be approximated by the standard singular monopole of charge λ_i over B_i. Since ∇ is a connection, for any central character $\chi \in Z_G^\vee$ the corresponding 2-form $\langle \chi, F_\nabla \rangle \in \Omega^2(Y')$ is well defined and closed. Thus, by Stokes' theorem, we have

$$0 = \int_{Y \setminus \bigcup_{i=1}^n B_i} d\langle \chi, F_\nabla \rangle = \sum_{i=1}^{n} \int_{\partial B_i} \langle \chi, F_\nabla \rangle.$$

Now, recall that

$$\int_{\partial B_j} \langle \chi, F_\nabla \rangle = -2\pi i \langle \chi, \lambda_j \rangle,$$

so it follows that $\sum_{i=1}^n \langle \chi, \lambda_i \rangle = 0$. □

Remark 3.2.7. Compare the previous proposition with Proposition 2.1.6.

When (∇, Φ) is in fact a $\mathfrak{h}$-monopole, solving the HB equation for some $C \in Z(\mathfrak{k})$, then another restriction arises relating the topology of the bundle with the location and weights of the singularities. We first give an auxiliary definition.

Definition 3.2.8. For any $\mathfrak{h}$-connection ∇ on $\mathbb{E}$ and any central character $\chi \in Z_G^\vee$, we define the χ-*Chern form of* ∇ as

$$c_1^\chi(\nabla) = \frac{i}{2\pi}\langle \chi, F_\nabla \rangle.$$

The χ-*degree of* ∇ is the integral

$$\deg_\chi(\nabla) = \frac{1}{2\pi}\int_{Y'} c_1^\chi(\nabla) \wedge dt.$$

For pairs with Dirac-type singularities, the χ-degree can be shown to be a topological invariant depending just on the topology of the bundle and the weights and locations of the singularities.

Proposition 3.2.9. *If* (∇, Φ) *is an* $\mathfrak{h}$*-pair on* $\mathbb{E}$ *with singularities of Dirac type* $(\mathbf{t}, \mathbf{x}, \boldsymbol{\lambda})$ *then, for any central character* χ*, the* χ*-degree of* ∇ *is equal to*

$$\deg_\chi(\nabla) = \deg(E_0(\chi)) - \frac{1}{2\pi}\sum_{i=1}^{n} t_i \langle \chi, \lambda_i \rangle,$$

for $E_0 \to X$ *the fibre* $E_0 = \mathbb{E}|_{\{0\}\times X}$ *and* E_0 *the complex line bundle associated to the action of* G *on* $\mathbb{C}^*$ *induced by the central character* χ.

Proof. To simplify the notation, we assume that all the t_i are different, but the same proof works for the general case. We want to compute the χ-degree

$$\deg_\chi(\nabla) = \frac{1}{2\pi}\int_{Y'} c_1^\chi(\nabla) \wedge dt = \frac{1}{2\pi}\frac{i}{2\pi}\int_{Y'} \langle \chi, F_\nabla \rangle \wedge dt.$$

For each $t \in (0, 2\pi)$, we denote

$$f_\chi(t) = \int_{\{t\}\times X} c_1^\chi(\nabla),$$

by putting $t_0 = 0$ and $t_{n+1} = 2\pi$, we get

$$\deg_\chi(\nabla) = \sum_{i=0}^{n}\int_{t_i}^{t_{i+1}} f_\chi(t)dt.$$

Now, by Stokes' theorem, for any point $y \in Y$, for a sufficiently small $\epsilon > 0$, and for any 2-form ξ on Y, we have

$$\int_{\{t+\epsilon\}\times X} \xi - \int_{\{t-\epsilon\}\times X} \xi - \int_{\partial B(y,\epsilon)} \xi = \int_{((t-\epsilon,t+\epsilon)\times X)\setminus B(y,\epsilon)} d\xi.$$

Therefore, for any $i = 1, \ldots, n$,

$$f_\chi(t_i + \epsilon) - f_\chi(t_i - \epsilon) - \int_{\partial B(y_i,\epsilon)} c_1^\chi(\nabla) = \int_{((t_i-\epsilon,t_i+\epsilon)\times X)\setminus B(y_i,\epsilon)} d(c_1^\chi(\nabla)).$$

Now, since $c_1^\chi(\nabla)$ is the Chern form of a connection, it is closed. On the other hand, since (∇, Φ) has a Dirac-type singularity of weight λ_i at y_i, we have

$$\int_{\partial B(y_i,\epsilon)} c_1^\chi(\nabla) = \frac{i}{2\pi}(-2\pi i\langle \chi, \lambda_i\rangle) = \langle \chi, \lambda_i\rangle.$$

Thus,

$$f_\chi(t_i + \epsilon) - f_\chi(t_i - \epsilon) = \langle \chi, \lambda_i\rangle.$$

We conclude that f_χ has the form

$$f_\chi(t) = f_\chi(0) + \sum_{i:t_i<t} \langle \chi, \lambda_i\rangle = \deg(E_0(\chi)) + \sum_{i:t_i<t} \langle \chi, \lambda_i\rangle.$$

Summing up, we get that the χ-degree of ∇ is

$$\deg_\chi(\nabla) = \frac{1}{2\pi}\sum_{i=0}^{n}(t_{i+1} - t_i)\left(\deg(E_0(\chi)) + \sum_{j=1}^{i}\langle \chi, \lambda_j\rangle\right).$$

Moreover, recall that $\sum_{i=1}^{n}\langle \chi, \lambda_i\rangle = 0$. Using this, we get

$$\deg_\chi(\nabla) = \deg(E_0(\chi)) - \frac{1}{2\pi}\sum_{i=1}^{n} t_i\langle \chi, \lambda_i\rangle.$$

As we wanted to show. □

Proposition 3.2.10. *If (∇, Φ) is a singular $\mathfrak{h}$-monopole on $\mathbb{E}$ of type $(\mathbf{t}, \mathbf{x}, \boldsymbol{\lambda})$, solving the HB equation for a constant $C \in Z(\mathfrak{k})$, then*

$$\langle \chi, C\rangle = -2\pi \deg_\chi(\nabla).$$

Proof. This this is an immediate consequence of the fact that, for any central character χ, we have $\int_{Y'}\langle \chi, *\nabla\Phi\rangle \wedge dt = 0$, which in turn follows from $*\nabla\Phi \wedge dt = \nabla_t(\Phi\omega_X)$. □

Special reductions on mini-holomorphic bundles

Upon further inspection of the HB equation $F_\nabla - *\nabla\Phi = iC\omega_X$, we note that (locally), this is an equation for $\mathfrak{g}$-valued 2-forms on Y and thus it splits into $(2,0)$, $(1,1)$ and $(0,2)$ parts. After some manipulations, we obtain the equations

$$\begin{cases} F_{\nabla,X} - *\nabla_t\Phi = iC\omega_X \\ [\nabla_X^{0,1}, \nabla_t - i\Phi(-)dt] = 0, \\ [\nabla_X^{1,0}, \nabla_t + i\Phi(-)dt] = 0. \end{cases}$$

The first of these equations is called the *real HB equation*, while the other two are called the *complex HB equations*. By the properties of $\mathfrak{h}$-connections, both complex equations are equivalent, and they give the integrability of the associated mini-holomorphic structure on $\mathbb{E}$. Reciprocally, if $\mathcal{E}$ is a mini-holomorphic bundle with underlying principal bundle $\mathbb{E}$, then its Chern pair (∇, Φ) is a solution to the HB equation if and only if it satisfies the real HB equation.

Definition 3.2.11. Let $\mathcal{E} \to Y$ be a singular mini-holomorphic G-bundle of type $(\mathbf{t}, \mathbf{x}, \boldsymbol{\lambda})$ and let h be a K-reduction of the underlying principal G-bundle. Let (∇, Φ) be the Chern pair associated to the K-reduction h.

We say that h is *Hermitian–Bogomolny* (HB) if (∇, Φ) is a solution to the HB equation and we say that it is *admissible* if (∇, Φ) has singularities of Dirac type $(\mathbf{t}, \mathbf{x}, \boldsymbol{\lambda})$. If h is both Hermitian–Bogomolny and admissible, we say that it is a *monopole* K*-reduction*.

It is clear that the existence of the special reductions defined above is subject to the topological restrictions. In particular, if $\mathcal{E}$ admits an admissible reduction, then $\sum_{i=1}^{n} \langle \chi, \lambda_i \rangle = 0$. Moreover, for any central character χ, we can define the χ*-degree of* $\mathcal{E}$ as

$$\deg_{\chi}(\mathcal{E}) = \deg_{\chi}(\nabla),$$

where ∇ is the Chern connection associated to any admissible reduction. This definition does not depend on the choice of admissible reduction, and in fact

$$\deg_{\chi}(\mathcal{E}) = \deg(E_0(\chi)) - \frac{1}{2\pi} \sum_{i=1}^{n} t_i \langle \chi, \lambda_i \rangle.$$

Once we assume the existence of an admissible reduction, it only makes sense to look for reductions h giving solutions to the HB equation with a fixed value of C, such that

$$\langle \chi, C \rangle = -2\pi \deg_{\chi}(\mathcal{E}),$$

for any central character χ.

Stability and the Charbonneau–Hurtubise–Smith correspondence

Let $P \subset G$ be a parabolic subgroup. Let $\mathcal{E} \to Y$ be a mini-holomorphic G-bundle and consider $\sigma : \mathcal{E} \to G/P$ a mini-holomorphic reduction of the structure group of $\mathcal{E}$ from G to P. That is, σ is compatible with the mini-holomorphic structure of $\mathcal{E}$ and with the holomorphic structure of G/P in such a way that the resulting P-bundle $\mathcal{E}_{\sigma} := \sigma^{-1}(P) \subset \mathcal{E}$ is also mini-holomorphic. If h is an admissible K-reduction, then the induced K-reduction on $\mathcal{E}_{\sigma}$ is also admissible. Indeed, being admissible is a condition which is just verified locally at the level of T-bundles.

To any parabolic subgroup $P \subset G$ we can associate a natural central character $\chi_P \in X^*(Z_P)$. Namely, we consider the Levi decomposition $P = LP^u$ and the ajdoint representation $\mathrm{Ad}_L^{\mathfrak{p}^u} : L \to GL(\mathfrak{p}^u)$. We now define the central character $\chi_P : L \to \mathbb{C}^*$ to be

$$\chi_P = \det \circ \mathrm{Ad}_L^{\mathfrak{p}^u}.$$

The following definitions of stability and polystability are due to Smith [Smi16] and to Elliott and Pestun [EP19], respectively.

Definition 3.2.12. Let $\mathcal{E}$ be a singular mini-holomorphic bundle on Y admitting an admissible K-reduction. We say that $\mathcal{E}$ is *stable* if for any maximal parabolic subgroup $P \subset G$ and for any mini-holomorphic reduction $\sigma : \mathcal{E} \to G/P$, we have that

$$\deg_{\chi_P}(\mathcal{E}_{\sigma}) < 0.$$

We say that $\mathcal{E}$ is *polystable* if there exists a (not necessarily maximal) parabolic subgroup $P \subset G$ with Levi factor L and a mini-holomorphic reduction of structure group $\sigma : \mathcal{E} \to G/L$ such that

1. the mini-holomorphic bundle $\mathcal{E}_\sigma$ is stable,
2. for every character $\chi \in X^*(P)$ which is trivial on Z_G, we have

$$\deg(E_{\sigma,0}(\chi)) = 0,$$

   for $E_{\sigma,0} = \mathcal{E}_\sigma|_{\{0\}\times X} \to X$.

We can now state the CHS correspondence.

Theorem 3.2.13 (Charbonneau–Hurtubise–Smith)**.** *A mini-holomorphic bundle $\mathcal{E}$ admits a monopole K-reduction if and only if it is polystable.*

Remark 3.2.14*.* The above theorem was originally proved by Charbonneau and Hurtubise [CH11] for the vector bundle case ($G = GL_n(\mathbb{C})$) and later generalized by Smith [Smi16] to the general case of any reductive group. The original result has been widely generalized in Mochizuki's book [Moc22]. The reader can also refer to the papers of Biswas and Hurtubise [BH15] and of Yoshino [Yos19] for other generalizations. We also refer to Elliott–Pestun [EP19, Section 6] for a survey of the topic.

Although the proof of the CHS correspondence is beyond the scope of this document, for completion we can give a very brief outline of the main ideas involved. The polystability condition follows from the existence of a monopole reduction by a standard argument, so really the difficult part is the other direction. The proof relies in two essential facts. The first one is that by using the results of Kronheimer [Kro85] and Pauly [Pau98] one can locally resolve Dirac type singularities by pulling back through the Hopf fibration. Using this, one can glue appropiately to obtain a holomorphic G-bundle on the 4-manifold $S^1 \times S^1 \times X$ and a K-reduction on it with suitable regularity. The second essential fact is the Hitchin–Kobayashi correspondence between polystable holomorphic bundles and instantons, which at this level of generality is due to Simpson. This yields an instanton on $S^1 \times S^1 \times X$ which after dimensionally reducing to $S^1 \times X$ gives a solution to the HB equation.

3.3 THE MODULI SPACE OF MONOPOLES

The moduli spaces of magnetic monopoles in $\mathbb{R}^3$ and their natural metrics have been widely studied in the gauge theory literature, for example in the book of Atiyah and Hitchin [AH88] and in the paper by Cherkis and Kapustin [CK02]. Using similar methods to the ones used in those texts, we can formally construct a moduli space of singular monopoles on the product of a Riemann surface and a circle, with prescribed singularities. Again, we repeat that the constructions that we reproduce here are valid at a formal level, whereas a rigorous treatment of the problem would involve several analytic considerations, the study of which we omit in this thesis. For a lot of our concerns, we follow the arguments of Elliott and Pestun [EP19].

The moduli space of singular monopoles is naturally equipped with a Kähler structure and moreover, when the Riemann surface has genus 1, it admits a hyper-Kähler structure. In that case, the moduli space can be constructed equivalently as a holomorphic-symplectic quotient, yielding a holomorphic-symplectic form equivalent via the CHS correspondence to the Hurtubise–Markman symplectic form on the moduli space of simple multiplicative Higgs bundles.

For the following, we repeat the setting of the previous section. We recall that X is a compact Riemann surface with Kähler form ω_X normalized to 1 and put $Y = S^1 \times X$. We also let G be a complex reductive group and fix a maximal torus $T \subset G$, a Borel subgroup $B \subset G$ and a maximal compact subgroup $K \subset G$ compatible with the pair (B, T) in the sense that T is a complexification of a maximal torus of K (in other words, T is $\mathbb{R}$-split for the real form K).

We prescribe the singularities by taking $\mathbf{x} = (x_1, \dots, x_n)$ a tuple of points of X, assumed to be distinct, $\mathbf{t} = (t_1, \dots, t_n)$ a tuple of angles in $(0, 2\pi)$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ a tuple of dominant cocharacters $\lambda_i \in X_*(T)_+$. We put $y_i = (e^{it_i}, x_i)$, $\mathbf{y} = (y_1, \dots, y_n)$, $Y' = Y \setminus \{y_1, \dots, y_n\}$ and $X' = X \setminus \{x_1, \dots, x_n\}$.

The Kähler quotient

Let $\mathbb{E} \to Y'$ be a principal G-bundle and $h : \mathbb{E} \to G/K$ a K-reduction of $\mathbb{E}$. Let $\mathcal{C}$ be the space of h-pairs (∇, Φ) with singularities of Dirac type $(\mathbf{t}, \mathbf{x}, \boldsymbol{\lambda})$. This is an infinite dimensional affine space modelled over the vector space

$$\Omega^1(Y', \mathbb{E}_h(\mathfrak{k})) \oplus \Omega^0(Y', \mathbb{E}_h(\mathfrak{k}))$$

and it admits a natural symplectic form ω_t defined as follows. Take a point $(\nabla, \Phi) \in \mathcal{C}$ and consider two vectors $(B_1, \Psi_1), (B_2, \Psi_2) \in T_{(A,\Phi)}\mathcal{C}$. Now, recall that we have an splitting

$$\Omega^1(Y', \mathbb{E}_h(\mathfrak{k})) = \mathrm{pr}_X^* \, \Omega^1(X', \mathbb{E}_h(\mathfrak{k})) \oplus \Omega^0(Y', \mathbb{E}_h(\mathfrak{k}))dt,$$

so we can decompose $B_i = B_{i,X} + B_{i,t}dt$. Let us choose now an invariant bilinear form κ on $\mathfrak{k}$, which naturally induces a map $\kappa : \Omega^2(X', \mathbb{E}_h(\mathfrak{k})) \to \Omega^2(X')$, and define

$$\omega_t((B_1, \Psi_1), (B_2, \Psi_2)) = \int_{Y'} (\kappa(B_{1,t}, \Psi_2) - \kappa(\Psi_1, B_{2,t}))\omega_X \wedge dt - \kappa(B_{1,X} \wedge B_{2,X})dt.$$

Here, we recall that $\wedge : \Omega^1(X', \mathbb{E}_h(\mathfrak{k}))^2 \to \Omega^2(X', \mathbb{E}_h(\mathfrak{k}))$ takes the wedge product on the form part and the Lie bracket on the Lie algebra part. The form ω_t is clearly closed since it does not depend on the pair (∇, Φ).

Consider now the *group of gauge transformations* $\mathcal{G} = \Omega^0(Y', \mathbb{E}_h(K))$, which acts on $\mathcal{C}$ by preserving the symplectic form ω_t, since the form is chosen to be Ad-invariant. A standard computation shows that the moment map for this action is

$$\begin{aligned} \mu_t : \mathcal{C} &\longrightarrow \Omega^2(Y', \mathbb{E}_h(\mathfrak{k})) \\ (\nabla, \Phi) &\longmapsto F_{\nabla,X} - *\nabla_t \Phi, \end{aligned}$$

Note that $\mu_t(\nabla, \Phi)$ is precisely the left hand side of the real HB equation. We can then consider the symplectic quotient $\mu_t^{-1}(iC\omega_X)/\mathcal{G}$ and get the space of monopoles by further imposing the complex equations.

Definition 3.3.1. The *moduli space of singular* $\mathfrak{h}$*-monopoles on* $\mathbb{E}$ *of type* $(\mathbf{t}, \mathbf{x}, \boldsymbol{\lambda})$ is the subvariety

$$\mathcal{M}_{\mathbf{t},\mathbf{x},\boldsymbol{\lambda}} = \mathcal{M}_{\mathbf{t},\mathbf{x},\boldsymbol{\lambda}}(\mathbb{E}, \mathfrak{h}) = \left\{[A, \Phi] \in \mu_t^{-1}(iC\omega_X)/G : [\nabla_X^{0,1}, \nabla_t - i\Phi(-)dt] = 0\right\}.$$

The space $\mathcal{C}$ is naturally endowed with a Riemannian metric

$$g((B_1, \Psi_1), (B_2, \Psi_2)) = \int_{Y'} \kappa(B_{1,X}, B_{2,X}) \wedge dt + \kappa(B_{1,t}, B_{2,t})\omega_X \wedge dt + \kappa(\Psi_1, \Psi_2)\omega_X \wedge dt.$$

Now, if we consider the complex structure I_t on $\mathcal{C}$ defined as $I_t(B_X) = - * B_X$, $I_t(B_t) = \Psi$ and $I_t(\Psi) = -B_t$, it follows that

$$\omega_t((B_1, \Psi_1), (B_2, \Psi_2)) = g((B_1, \Psi_1), I_t(B_2, \Psi_2)).$$

Therefore, the triple (g, ω_t, I_t) determines a Kähler structure on $\mathcal{C}$. The symplectic quotient $\mu_t^{-1}(iC\omega_X)/\mathcal{G}$ can now be regarded as a Kähler reduction and thus it inherits a Kähler structure. Finally, since we have that the complex equation $[\nabla_X^{0,1}, \nabla_t - i\Phi(-)dt] = 0$ is I_t-holomorphic, we conclude that the space $\mathcal{M}_{\mathbf{t},\mathbf{x},\boldsymbol{\lambda}}$ has a natural Kähler structure induced from (g, ω_t, I).

The hyper-Kähler quotient

Suppose that the Riemann surface X has genus 1, and thus it has trivial canonical bundle $K_X \cong \mathcal{O}_X$. In that case, we can take local coordinates real (x, y) on X and the complex coordinate $z = x + iy$, and obtain global decompositions

$$\begin{aligned}\Omega^1(Y, \mathbb{C}) &= C^\infty(Y, \mathbb{C})dz \oplus C^\infty(Y, \mathbb{C})d\bar{z} \oplus C^\infty(Y, \mathbb{C})dt \\ &= C^\infty(Y, \mathbb{C})dx \oplus C^\infty(Y, \mathbb{C})dy \oplus C^\infty(Y, \mathbb{C})dt.\end{aligned}$$

Therefore, we can identify an element (∇, Φ) of $\mathcal{C}$ with a tuple (A_x, A_y, A_t, Φ), for the A_ν, $\nu = x, y, t$ and Φ sections of $\mathbb{E}_\mathfrak{h}(\mathfrak{k}) \to Y'$, so that we recover the covariant derivative of ∇ as

$$d_\nabla = d_A = d + A_x dx + A_y dy + A_t dt.$$

We denote $d_{A,\nu} = d + A_\nu d\nu$, for $\nu = x, y, t$. We can also define $\partial_{A,X} = d_{A,x} - i d_{A,y}$ and $\bar{\partial}_{A,X} = d_{A,x} + i d_{A,y}$.

In this case we can write

$$\begin{aligned}&\omega_t((B_1, \Psi_1),(B_2, \Psi_2)) \\ &\qquad = \int_{Y'} [\kappa(B_{1,t}, \Psi_2) - \kappa(\Psi_1, B_{2,t}) - (\kappa(B_{1,x}, B_{2,y}) - \kappa(B_{1,y}, B_{2,x}))] dx dy dt\end{aligned}$$

and it is then possible to define other two symplectic structures

$$\begin{aligned}
&\omega_x((B_1,\Psi_1),(B_2,\Psi_2)) \\
&\qquad = \int_{Y'} [\kappa(B_{1,x},\Psi_2) - \kappa(\Psi_1,B_{2,x}) - (\kappa(B_{1,y},B_{2,t}) - \kappa(B_{1,t},B_{2,y}))]dxdydt, \\
&\omega_y((B_1,\Psi_1),(B_2,\Psi_2)) \\
&\qquad = \int_{Y'} [\kappa(B_{1,y},\Psi_2) - \kappa(\Psi_1,B_{2,y}) - (\kappa(B_{1,t},B_{2,x}) - \kappa(B_{1,x},B_{2,t}))]dxdydt.
\end{aligned}$$

We also obtain a more explicit description of the complex structre I_t as

$$I_t(B_x,B_y,B_t,\Psi) = (B_y,-B_x,\Psi,-B_t)$$

and we can thus define other two complex structures

$$\begin{aligned}
I_x(B_x,B_y,B_t,\Psi) &= (\Psi,B_t,-B_y,-B_x) \\
I_y(B_x,B_y,B_t,\Psi) &= (B_t,-\Psi,-B_x,B_y).
\end{aligned}$$

One checks easily that these complex structures satisfy the quaternion relations

$$\begin{cases} I_x^2 = I_y^2 = I_t^2 = -1 \\ I_xI_y = I_t \end{cases}$$

and that I_x and I_y are g-compatible with ω_x and ω_y, respectively. Therefore, we obtain a hyper-Kähler structure on the space $\mathcal{C}$.

The action of the group $\mathcal{G}$ is also symplectic for ω_x and ω_y and the resulting moment maps are, respectively

$$\begin{aligned}
\mu_x(\nabla,\Phi) &= d_{A_x}\Phi - [d_{A_y},d_{A_t}], \\
\mu_y(\nabla,\Phi) &= d_{A_y}\Phi - [d_{A_t},d_{A_x}].
\end{aligned}$$

These two equations can be obtained from the HB equation and can be combined to give the complex equation, which in this case can be written as

$$[\bar{\partial}_{A,X}, d_{A,t} - i\Phi(-)dt] = 0.$$

We can put these moment maps together to obtain a *hyper-Kähler moment map*

$$\begin{aligned}
\boldsymbol{\mu} : \mathcal{C} &\longrightarrow \mathrm{Lie}(\mathcal{G})^* \otimes \mathbb{R}^3 \\
(\nabla,\Phi) &\longmapsto F_\nabla - *d_\nabla\Phi,
\end{aligned}$$

and thus we obtain the moduli space of monopoles as the *hyper-Kähler quotient*

$$\mathcal{M}_{\mathbf{t},\mathbf{x},\boldsymbol{\lambda}} = \boldsymbol{\mu}^{-1}(iC\omega_X)/\mathcal{G}.$$

As a holomorphic-symplectic quotient

In the same setting as above, by putting ω_x and ω_y together we can obtain a holomorphic-symplectic form $\Omega = \omega_x + i\omega_y$. It is easy to check that this coincides with the Hurtubise–Markman symplectic form on the moduli space of multiplicative Higgs bundles. Moreover, the group of *complex gauge transformations* $\mathcal{G}^{\mathbb{C}} = \Omega^0(Y', \mathbb{E}(\mathfrak{g}))$ acts on $\mathcal{C}$ holomorphic-symplectically with holomorphic moment map given by

$$M(\nabla, \Phi) = [\bar{\partial}_{A,X}, d_{A,t} - i\Phi(-)dt].$$

One can then also obtain the moduli space $\mathcal{M}_{t,x,\lambda}$ as a holomorphic-symplectic manifold, by restricting to the subspace $\mathcal{C}^s \subset \mathcal{C}$ formed by pairs which define a stable multiplicative Higgs bundle, and taking the holomorphic-symplectic quotient $(M^{-1}(0) \cap \mathcal{C}^s)/\mathcal{G}^{\mathbb{C}}$.

3.4 INVOLUTIONS AND FIXED POINTS, II

Mini-holomorphic bundles and involutions

Let G be a complex reductive group and let $\theta \in \mathrm{Aut}_2(G)$ be a holomorphic involution of G. Let $\epsilon \in \{-1, 1\}$. We fix X, Y, x and t as in the previous section and keep our notations from there. Moreover, we fix a maximal θ-split torus $A \subset G$, a θ-stable maximal torus $T \subset G$ containing it, and a Borel subgroup $B \subset G$ contained in a minimal θ-split parabolic and with $T \subset B$.

To any singular mini-holomorphic G-bundle $\mathcal{E} \to Y$ we can associate another mini-holomorphic G-bundle

$$\iota^\theta_\epsilon(\mathcal{E}) = \zeta^*_\epsilon \theta(\mathcal{E}),$$

where $\theta(\mathcal{E})$ is the associated G-bundle to the action of G on itself by θ, and $\zeta_\epsilon : Y \to Y$ is the involution

$$\begin{aligned} \zeta_\epsilon : S^1 \times X &\longrightarrow S^1 \times X \\ (e^{it}, x) &\longmapsto (e^{\epsilon i t}, x). \end{aligned}$$

Let $E = E_0 \to X$ be the restricted bundle $\mathcal{E}|_{\{1\}\times X}$, endowed with the holomorphic structure coming from the mini-holomorphic structure of $\mathcal{E}$. If $\varphi : E|_{X'} \to E|_{X'}$ denotes the scattering map of $\mathcal{E}$, then the scattering map of $\zeta^*_\epsilon \mathcal{E}$ is equal to φ^{-1}. Moreover, restricting $\theta(\mathcal{E})$ to X yields the holomorphic bundle $\theta(E)$ and the scattering map of $\theta(\mathcal{E})$ is $\theta(\varphi)$. Summing up, we have that the mini-holomorphic bundle $\iota^\theta_\epsilon(\mathcal{E})$ corresponds to the multiplicative G-Higgs bundle

$$\iota^\theta_\epsilon(E, \varphi) = (\theta(E), \theta(\varphi)^\epsilon).$$

In Section 2.3 we studied the map $(E, \varphi) \mapsto (\theta(E), \theta(\varphi)^\epsilon)$ and obtained its fixed points in the case it made sense to do it. We study the same problem now from the perspective of mini-holomorphic bundles, by considering the map $\mathcal{E} \mapsto \zeta^*_\epsilon \theta(\mathcal{E})$.

As for multiplicative Higgs bundles, the case $\epsilon = -1$ is particularly interesting. For that reason, we restrict ourselves to the case $\epsilon = -1$, the results for the case $\epsilon = 1$ being sufficiently straightforward for us to omit them here. To simplify the notation, we put $\zeta = \zeta_{-1} : (e^{it}, x) \mapsto (e^{-it}, x)$.

We also remark that, as in the case of multiplicative Higgs bundles, if θ and θ' are two involutions that differ by an inner automorphism, meaning that there exists $\alpha \in \mathrm{Int}(G)$ such that $\theta' = \alpha \circ \theta$, then $\theta(\mathcal{E}) \cong \theta'(\mathcal{E})$. Therefore, for any class $a \in \mathrm{Out}_2(G)$, the map $\iota_-^a : \mathcal{E} \mapsto \zeta^*\theta(\mathcal{E})$ for any representative element θ of a, is well defined at the level of isomorphism classes of mini-holomorphic bundles.

A first important thing to note is that if $\mathcal{E}$ is of type $(\mathbf{t}, \mathbf{x}, \boldsymbol{\lambda})$, then $\zeta^*\theta(\mathcal{E})$ is of type $(\bar{\mathbf{t}}, \mathbf{x}, (-\theta(\boldsymbol{\lambda}))_+)$, for

$$\bar{\mathbf{t}} = (2\pi - t_n, 2\pi - t_{n-1}, \dots, 2\pi - t_1),$$

and where we recall that $(-\theta(\lambda_i))_+$ denoted the dominant cocharacter in the orbit of $-\theta(\lambda_i)$ under the Weyl group. Since we assumed the x_i to be distinct, we have $(\mathbf{t}, \mathbf{x}) = (\bar{\mathbf{t}}, \mathbf{x})$ if and only if $\mathbf{t} = (\pi, \pi, \dots, \pi)$. We denote this vector simply by $\mathbf{t} = \pi$. Therefore, a necessary condition for (the isomorphism class) of a mini-holomorphic G-bundle to be fixed under the involution ι_-^a is that it is of type $(\pi, \mathbf{x}, \boldsymbol{\lambda})$, with $\boldsymbol{\lambda}$ satisfying the condition $\boldsymbol{\lambda} = (-\theta(\boldsymbol{\lambda}))_+$.

As for multiplicative Higgs bundles, we can say that a mini-holomorphic bundle $\mathcal{E}$ is *simple* if its only mini-holomorphic automorphisms are those given by multiplying by elements of the centre Z_G of G. We can go on now to describe the simple mini-holomorphic bundles which are fixed under the involution ι_-^a, but before we need an auxiliary definition.

Definition 3.4.1. Let $c \in Z_G^\theta$ be an element of the centre of G fixed under θ. A $(\mathbb{Z}/2, \theta, \zeta, c)$-*equivariant structure* on $\mathcal{E}$ is a diffeomorphism η of $\mathcal{E}$ preserving the mini-holomorphic structure such that

1. the following diagram commutes

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\eta} & \mathcal{E} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\zeta} & Y, \end{array}$$

2. for every $g \in G$, we have $\eta(e \cdot g) = e \cdot \theta(g)$,

3. and such that $\eta^2(e) = e \cdot c$.

Remark 3.4.2. The kind of structures just defined are a particular case of $(\Gamma, \theta, \zeta, c)$-equivariant structures, where Γ is a finite group, θ and ζ are homomorphisms from Γ to $\mathrm{Aut}(G)$ and to $\mathrm{Diffeo}(\mathcal{E})$, respectively, and $c \in Z^2(\Gamma, Z_G)$ is a 2-cocycle in the group cohomology theory of Γ with values in Z_G associated to the action induced by θ. We refer the reader to the paper [BGPGMiR23] for more information about this kind of equivariant structures.

Proposition 3.4.3. *If $\mathcal{E}$ is a simple singular mini-holomorphic bundle of type (π, x, λ) with $\lambda = (-\theta(\lambda))_+$, then $\mathcal{E} \cong \iota_-^a(\mathcal{E})$ if and only if it admits a $(\mathbb{Z}/2, \theta, \zeta, c)$-equivariant structure.*

Proof. Indeed, an isomorphism $E \longrightarrow \zeta^*\theta(E)$ can be understood as a diffeomorphism $\eta : E \longrightarrow E$ preserving the holomorphic structure and satisfying properties 1 and 2 in the definition of (θ, ζ, c)-equivariant structure. Now, from these properties it follows that η^2 is an automorphism of E, and thus, since E is simple, $\eta^2(e) = e \cdot c$ for some $c \in Z_G$. Moreover, if $f_\eta(e) \in G$ is such that $\eta(e) = e \cdot f_\eta(e)$, then $c = f_\eta(e)\theta(f_\eta(e))$ and thus $c = \theta(c)$. □

We can now recover our results on fixed points of multiplicative Higgs bundles from the point of view of mini-holomorphic bundles. Indeed, suppose that $\mathcal{E}$ admits a $(\mathbb{Z}/2, \theta, \zeta, c)$-equivariant structure given by η. Let $E_0 \longrightarrow X$ and $E_\pi \longrightarrow X$ denote the bundles $\mathcal{E}|_{\{1\}\times X}$ and $\mathcal{E}|_{\{-1\}\times X'}$ endowed with their respective holomorphic structures induced from $\mathcal{E}$. Both $\{1\} \times X$ and $\{-1\} \times X$ are submanifolds of Y fixed under ζ and thus η induces two biholomorphisms $\eta_0 = \eta|_{E_0}$ and $\eta_\pi = \eta|_{E_\pi}$. Moreover, from property 2 in the definition of equivariant structure, it follows that they are in fact θ-twisted automorphisms and from 3 it follows that the corresponding G-equivariant maps $f_{\eta_0} : E_0 \longrightarrow G$ and $f_{\eta_\pi} : E_\pi \longrightarrow G$ in factor through the subset

$$S_c^\theta = \{s \in G : s\theta(s) = c\}.$$

Since the maps f_{η_0} and f_{η_π} are G-equivariant, they must factor through θ-twisted orbits $G * s_0$ and $G * s_\pi$ respectively, for $s_0, s_\pi \in S_c^\theta$. These orbits are isomorphic respectively to G/G^{θ_0} and G/G^{θ_π}, for $\theta_0 = \mathrm{Int}_{s_0} \circ \theta$ and $\theta_\pi = \mathrm{Int}_{s_\pi} \circ \theta$. We have shown the following.

Proposition 3.4.4. *If (the isomorphism class of) $\mathcal{E}$ is fixed under ι_-^a, then there exist two involutions θ_0, θ_π representing the class a such that E_0 reduces to G^{θ_0} and E_π reduces to G^{θ_π}.*

The reduction of the scattering map φ that we gave in Theorem 2.3.4 can be now deduced from the fact that $\eta_\pi = \eta_0 \circ \varphi$. Indeed, from here and using that $\eta_\pi^2(e) = e \cdot c$ for any $e \in E_\pi$, we obtain that

$$f_{\eta_0}(e)\theta(f_\varphi(e))f_{\eta_0}(e)^{-1} = f_\varphi(e)^{-1},$$

so if $f_{\eta_0}(e) = s_0$, we have that f_φ restricted to the reduction to G^{θ_0} factors through S^{θ_0}. The section φ can then be interpreted as measuring the difference between the two reductions to G^{θ_0} and G^{θ_π} and, indeed, $G^{\theta_0} = G^{\theta_\pi}$ if and only if s_0 and s_π are in the same θ-twisted G-orbit, which happens if and only if f_φ takes values in M^{θ_0}. We conclude the following.

Proposition 3.4.5. *A mini-holomorphic bundle $\mathcal{E}$ corresponds to a multiplicative G-Higgs bundle coming from a multiplicative (G, θ)-Higgs bundle if and only if there exists some θ' in the same class in $\mathrm{Out}_2(G)$ as θ and some $c \in Z_G^{\theta'}$ such that E has a $(\mathbb{Z}/2, \theta', \zeta, c)$-equivariant structure with f_{η_0} and f_{η_π} taking values in the same θ'-twisted G-orbit, isomorphic to G/G^θ.*

Singular monopoles and involutions

Let $K \subset G$ be a θ-stable maximal compact subgroup of G and let h be a K-reduction of a singular mini-holomorphic G-bundle $\mathcal{E}$ on Y. Since K is θ-stable, for $\epsilon \in \{-1, 1\}$ the map $\zeta_\epsilon^*\theta(h) : \zeta_\epsilon^*\theta(\mathcal{E}) \to G/K$ is also G-equivariant and thus it also defines a K-reduction. Moreover, if (∇, Φ) is the Chern pair associated to h, then the Chern pair associated to $\zeta_\epsilon^*\theta(h)$ is equal to $\zeta_\epsilon^*\theta_*(\nabla, \Phi)$. Locally, if ∇ is determined by a $\mathfrak{k}$-valued 1-form A that we may decompose as $A = A_X + A_t$, we have

$$\zeta_\epsilon^*\theta_*(A_X, A_t, \Phi) = (\theta(A_X(e^{\epsilon i t}, x)), \epsilon\theta(A_t(e^{\epsilon i t}, x)), \epsilon\theta(\Phi(e^{\epsilon i t}, x))).$$

Since we are assuming that $\mathbf{t} = \pi$, for any central character χ we have

$$\deg^\chi(\mathcal{E}) = \deg(E_0(\chi)).$$

Using this it is easy to show that if a mini-holomorphic bundle $\mathcal{E}$ is polystable then $\zeta_\epsilon^*\theta(\mathcal{E})$ is also polystable, and thus the involution ι_ϵ^a induces the following involution on the moduli space of h-monopoles

$$\begin{aligned} \mathcal{M}_{\pi,\boldsymbol{x},\boldsymbol{\lambda}} &\longrightarrow \mathcal{M}_{\pi,\boldsymbol{x},\boldsymbol{\lambda}} \\ (A_X, A_t, \Phi) &\longmapsto (\theta(A_X(e^{\epsilon i t}, x)), \epsilon\theta(A_t(e^{\epsilon i t}, x)), \epsilon\theta(\Phi(e^{\epsilon i t}, x))). \end{aligned}$$

A straightforward computation shows that, indeed, the h-pair on the right hand side verifies the HB equation, provided that the one on the left hand side does.

As a consequence, for an h-monopole fixed under this involution, we have that the tuples $(A_X, A_t, \Phi)|_{\{1\}\times X}$ and $(A_X, A_t, \Phi)|_{\{-1\}\times X}$ are gauge equivalent to $(\theta(A_X), \epsilon\theta(A_t), \epsilon\theta(\Phi))|_{\{1\}\times X}$ and $(\theta(A_X), \epsilon\theta(A_t), \epsilon\theta(\Phi))|_{\{-1\}\times X}$, respectively. So there exist two involutions θ_0 and θ_π representing the same class in $\mathrm{Out}_2(G)$ as θ such that $A_X|_{\{1\}\times X}$ takes values in $\mathfrak{k}^{\theta_0}$, $A_t|_{\{1\}\times X}$ and $\Phi|_{\{1\}\times X}$ take values in $\mathfrak{m}_K^{\theta_0}$, $A_X|_{\{-1\}\times X}$ takes values in $\mathfrak{k}^{\theta_\pi}$, and $A_t|_{\{1\}\times X}$ and $\Phi|_{\{1\}\times X}$ take values in $\mathfrak{m}_K^{\theta_\pi}$. Here, for any involution θ, if K is a θ-stable maximal compact subgroup and we consider the restriction of θ to K, then $\mathfrak{k}^\theta$ and $\mathfrak{m}_K^\theta$ are respectively the $+1$ and -1 eigenspaces appearing in the Cartan decomposition

$$\mathfrak{k} = \mathfrak{k}^\theta \oplus \mathfrak{m}_K^\theta.$$

When we assume that X has genus 1 we can show how the involutions ι_ϵ^a behave with respect to the hyper-Kähler structure on the moduli space $\mathcal{M}_{\pi,\boldsymbol{x},\boldsymbol{\lambda}}$. This case is of particular interest since special submanifolds of hyper-Kähler manifolds appear in the context of mirror symmetry. In particular, a manifold which is Lagrangian with respect to one of the symplectic structures ω_ν of the hyper-Kähler manifold is said to be of A type with respect to that component, whereas if it is a complex submanifold with respect to the structure I_ν, then it is said to be of B type. This way, by a submanifold of type (B, B, B) we mean a hyper-Kähler submanifold. On the otherhand, by a submanifold of type (B, A, A) we mean a submanifold which is complex with respect to the first component but Lagrangian with respect to the other two. Analogously, we can say that a diffeomorphism of a hyper-Kähler manifold into itself is of A type with respect to a certain component if it is Lagrangian or, equivalently, anti-holomorphic, with respect to that component and of B type if it is holomorphic or, equivalently, symplectic. If a map is of A type (respectively, of B type) with respect to a component, then its fixed points form a submanifold of A type (respectively, of B type) with respect to that component.

Recall that ι_a^ϵ acts on a tuple (A_x, A_y, A_t, Φ) by mapping it to the tuple

$$(\theta(A_x(\epsilon t, x, y)), \theta(A_y(\epsilon t, x, y)), \epsilon\theta(A_t(\epsilon t, x, y)), \epsilon\theta(\Phi(\epsilon t, x, y))).$$

Therefore, the differential of ι_a^ϵ acts as

$$(B_x, B_y, B_t, \Psi) \mapsto (\epsilon\theta(B_x), \epsilon\theta(B_y), \theta(B_t), \theta(\Psi)).$$

Thus,

$$\iota_{a,*}^\epsilon \circ (I_t, I_x, I_y) = (I_t, \epsilon I_x, \epsilon I_y) \circ \iota_{a,*}^\epsilon$$

and

$$(\iota_a^\epsilon)^*(\omega_t, \omega_x, \omega_y) = (\omega_t, \epsilon\omega_x, \epsilon\omega_y).$$

We conclude that ι_a^ϵ is of type (B, B, B) when $\epsilon = 1$ and of type (B, A, A) when $\epsilon = -1$.

FURTHER DIRECTIONS

LANGLANDS DUALITY AND MIRROR SYMMETRY

We expect that multiplicative Hitchin fibrations are Langlands dual in a similar manner as how "additive" Hitchin fibrations are, by the results of Donagi and Pantev [DP12]. More precisely, in forthcoming joint work with Benedict Morrissey [GM], we study the following problem. We assume that G is semisimple and let G^{sc} be its simply-connected cover. Let $\check{G}$ be the Langlands dual group of G and $\check{G}^{sc}$ its simply-connected cover (which is isomorphic to the Langlands dual group of G^{ad}). Consider $\mathscr{M}(G, G^{sc})$ to be the moduli stack of G^{sc}-valued multiplicative G-Higgs bundles, by which we mean pairs (E, φ) with $E \to X$ a principal G-bundle and φ a meromorphic section of $E(G^{sc})$, which is well defined since G acts on G^{sc} through the adjoint action. We define the moduli stack $\mathscr{M}(\check{G}, \check{G}^{sc})$ in a similar way. We remark that it follows from considering the short exact sequence $1 \to \pi_1(G) \to G^{sc} \to G \to 1$ and the long exact sequence associated to it both of these stacks are just connected components of the usual moduli stacks of multiplicative Higgs bundles $\mathscr{M}(G)$ and $\mathscr{M}(\check{G})$ where some topological invariant is fixed (see [HM02, Section 3.2]). We can thus consider the Hitchin bases $\mathscr{B}(G, G^{sc})$ and $\mathscr{B}(\check{G}, \check{G}^{sc})$ associated to these stacks, and the corresponding multiplicative Hitchin fibrations h_G and $h_{\check{G}}$.

Note that the invariant $\mathrm{inv}(\varphi)$ for a pair (E, φ) in $\mathscr{M}(G, G^{sc})$ is a divisor with values in $X_*(T^{sc})$, which is the coroot lattice of G. The vector space isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$ induced by an invariant bilinear form sends coroots to roots, and thus matches the coroot lattice $X_*(T^{sc})$ of G with the root lattice $X^*(T^{ad})$ which is equal to $X_*(\check{T}^{sc})$, the coroot lattice of $\check{G}$. Therefore, the corresponding stacks of invariants $\mathscr{A}(G, G^{sc})$ and $\mathscr{A}(\check{G}, \check{G}^{sc})$ are matched under Langlands duality. This motivates the conjecture.

Conjecture 1. *The multiplicative Hitchin maps* $h_G : \mathscr{M}(G, G^{sc}) \to \mathscr{B}(G, G^{sc})$ *and* $h_{\check{G}} : \mathscr{M}(\check{G}, \check{G}^{sc}) \to \mathscr{B}(\check{G}, \check{G}^{sc})$ *are Langlands dual in the sense of Donagi–Pantev.*

This conjecture can be generalized to other stacks of multiplicative Higgs bundles of the form $\mathscr{M}(G_0, G_1)$, for G_0 and G_1 a pair of isogenous semisimple groups. Potentially, the assumption of semisimplicity can also be dropped. Note that under Langlands duality the fundamental group $\pi_1(G_1)$ is matched to its dual group $\pi_1(G_1)^\vee = Z_{\check{G}_1}$. Therefore, we conjecture the following.

Conjecture 2. *The multiplicative Hitchin maps* $h_G : \mathscr{M}(G_0, G_1) \to \mathscr{B}(G_0, G_1)$ *and* $h_{\check{G}} : \mathscr{M}(\check{G}_0, \check{G}_1^{sc}/Z_{\check{G}_1}) \to \mathscr{B}(\check{G}_0, \check{G}_1^{sc}/Z_{\check{G}_1})$ *are Langlands dual in the sense of Donagi–Pantev.*

The justification behind these conjectures relies on the well known properties of the multiplicative Hitchin fibration, studied by Bouthier, J. Chi and G. Wang [Bou15, BC18, Bou17, Chi22, Wan23]. In particular, the description of the regular centralizer for very flat monoids, as studied in the works of Chi and Wang, provides the evidence behind the matching of the two fibrations. Once the regular centralizers are matched, one can match the corresponding Picard stacks, and proceed as in the Donagi–Pantev case. More precisely, we believe that the general arguments given by Chen and Zhu [CZ17] in their proof and generalization of the result of Donagi–Pantev should translate to this case in a very straightforward manner.

Recall that it is conjectured that Higgs bundles for a real form $G_{\mathbb{R}}$ of G define the support of a (B, A, A)-brane in the Hitchin moduli space which corresponds to Higgs bundles for the Nadler dual group $\check{G}_{G_{\mathbb{R}}} \subset \check{G}$ of the real form $G_{\mathbb{R}}$ [GW09]. In the multiplicative case, we recall that, assuming that X is an elliptic curve, our Theorem 2.3.10 aswell as our comments at the end of Section 3.4, imply that (the image of) multiplicative (G, θ)-Higgs bundles form a submanifold of (B, A, A)-type inside the moduli space of multiplicative G-Higgs bundles. Therefore, we have a similar conjecture for this multiplicative situation, which we can roughly state as follows.

Conjecture 3. *Suppose that X is an elliptic curve. Then, for any $\theta \in \mathrm{Aut}_2(G)$, multiplicative (G, θ)-Higgs bundles define the support of a (B, A, A)-brane inside $\mathcal{M}(G, G^{sc})$ dual to the (B, B, B)-brane inside $\mathcal{M}(\check{G}, \check{G}^{sc})$ formed by multiplicative $\check{G}_{G/G^\theta}$-Higgs bundles, where $\check{G}_{G/G^\theta}$ is the dual group of G/G^θ.*

Some justification behind this conjecture is that by construction the dual group contains $\check{A}$, the dual of a maximal θ-split torus, as maximal torus, its Weyl group is W_θ and its dominant Weyl chamber is $X_*(A_{G^\theta})_+$. Hence, one should be able to match the corresponding stacks of invariants in each case.

REGULAR QUOTIENTS FOR SYMMETRIC VARIETIES

The description of the Hitchin fibration given by Ngô relies deeply in the descent argument [Ng10, Lemme 2.1.1] for the centralizer group scheme of the Lie algebra $\mathfrak{g}$ from the regular locus $\mathfrak{g}^{reg} \subset \mathfrak{g}$ to the GIT quotient $\mathfrak{g} /\!\!/ G$. The same problem can be formulated in the context of the *generalized Hitchin map* of Morrissey and Ngô [MN, Ng23] that we explain in Section 2.1.

We recall that the generalized Hitchin map is formulated by taking an affine variety Σ acted on by two reductive groups H and Z with commuting actions, and studying the map of stacks

$$[\Sigma/(H \times Z)] \longrightarrow [(\Sigma /\!\!/ H)/Z].$$

The analog of the centralizer group scheme is the group scheme of stabilizers $M \to \Sigma$ with

$$I_x = \{g \in G : g \cdot x = x\},$$

for $x \in \Sigma$, and consider the regular locus $\Sigma^{\mathrm{reg}} \subset \Sigma$ consisting of the points where the stabilizer has minimal dimension, so I is flat over Σ^{reg}. Ideally, one would want I to descend to a group scheme J (or more generally to a band in the sense of Giraud [Gir71] if I is not abelian) defined over the GIT quotient $\Sigma /\!\!/ H$, and show that $[\Sigma/H] \to \Sigma /\!\!/ H$ determines a gerbe banded by J . However, there are no-go results for this. Roughly, if the GIT quotient does not parametrize regular orbits, then I does not descend to a band. Examples of this behaviour are provided by the Hitchin fibration for symmetric pairs; the reader may consult the paper of García-Prada and Peón-Nieto [GPPN23, Example 1].

A solution for this consists in considering some intermediate space $\Sigma /\!\!/ H$, between the quotient stack $[\Sigma/H]$ and the GIT quotient $\Sigma /\!\!/ H$, parametrizing regular orbits, which is called the *regular quotient*. The definition of the regular quotient is due to Morrissey and Ngô [MN, Ng23], although similar ideas were originally proposed independently for the symmetric pair case by García-Prada and Peón-Nieto [GPPN23]. The construction of the regular quotient is based in the process of *rigidification* (see [AOV08, Appendix A]).

It is a result of Morrissey–Ngô [MN, Ng23] that the group scheme $I \to \Sigma^{\mathrm{reg}}$ descends to a band J over the regular quotient $\Sigma^{\mathrm{reg}} /\!\!/ H$ and the quotient stack $[\Sigma/H]$ is a gerbe banded by J. A Z-equivariant version of this is also given by Morrissey–Ngô, which allows to extend the results to the map $[\Sigma/(H \times Z)] \to [(\Sigma /\!\!/ H)/Z]$. It follows that the study of the generalized Hitchin map for this situation can be essentially reduced to the description of the regular quotient $\Sigma^{\mathrm{reg}} /\!\!/ H$.

Thus, in relation with the multiplicative Hitchin map for symmetric embeddings considered in Section 2.2 we suggest the consideration of the following problem.

Problem 1. *Let* G *be a semisimple simply-connected complex group and* $\theta \in \mathrm{Aut}_2(G)$ *an involution. Study the regular quotients*

$$(G/G^\theta) /\!\!/ G^\theta \quad \textit{and} \quad \mathrm{Env}(G/G^\theta) /\!\!/ G^\theta.$$

On the one hand, if $\mathfrak{m}^\theta$ denotes the -1-eigenspace of θ in $\mathfrak{g}$, the study regular quotient $\mathfrak{m}^\theta /\!\!/ G^\theta$ is the fundamental piece for the description of the (additive) Hitchin fibration for symmetric pairs. A complete description of this regular quotient is given in forthcoming work of Morrissey and Hameister [HM, Ham23]. On the other hand, descriptions of the regular quotients $G /\!\!/ G$ and $\mathrm{Env}(G) /\!\!/ G$ are provided in the work of Morrissey and Ngô.

Both the description of $\mathfrak{m}^\theta /\!\!/ G^\theta$ and of $G /\!\!/ G$ rely on the existence of sections or cross-sections of the GIT quotient $\Sigma \to \Sigma /\!\!/ H$. In the first case, the existence of such a section is a result of Kostant–Rallis [KR71], while in the group case there is the Steinberg cross-section [Ste65]. The existence of a similar cross-section for the global case of a symmetric variety G/G^θ acted on by G^θ remains, to our best knowledge, an open question. A positive answer to it would enable a method towards the study of Problem 1.

Question 1. *Is there a cross-section of the GIT quotient* $G/G^\theta \to (G/G^\theta) /\!\!/ G^\theta$*?*

GENERALIZATION TO SPHERICAL VARIETIES

The definition of a multiplicative (G, θ)-Higgs bundle can be generalized directly to the case of a pair (G, H), for any subgroup $H \subset G$. Indeed, one can consider pairs (E, φ) with $E \to X$ a principal H-bundle and φ a section of the associated bundle of symmetric spaces $E(G/H)$ through the left multiplication action of H on G/H, defined over the complement of a finite subset of X.

The invariant of such a φ is a divisor which takes values literally in the quotient of loop spaces $(G/H)(F)/G(\mathcal{O})$. It is a result of Luna and Vust [LV83] that this quotient is identified with a subset $\mathcal{V}^1$ of the *valuation cone* $\mathcal{V}$ of G/H. The valuation cone admits a natural injection inside the dual of the weight lattice, $P(G/H)^\vee$. We refer to Timashev's book [Tim11] for a definition of the valuation cone and for proofs of these results. Moreover, if we further assume that G/H is a spherical G-variety, then $\mathcal{V}^1 = \mathcal{V}$. If we fix a maximal torus T, recall that we can associate the torus $T_H = T/(T \cap H)$ to the spherical homogeneous space G/H and the weight lattice of G/H is equal to $P(G/H) = X^*(T_H)$. We refer to Section 1.4 for more details. Therefore, the set $(G/H)(F)/G(\mathcal{O})$ consists of cocharacters of the torus T_H. When G/H is a symmetric variety, we also know that these cocharacters are precisely the anti-dominant ones (that is, those lying in the intersection of $X_*(T_H)$ with the anti-dominant Weyl chamber).

It now makes sense to pose the following question.

Question 2. *Is there a generalization of the theory of very flat monoids and the theory of very flat symmetric embeddings to a theory of "very flat spherical varieties"? Is there a generalization of the Vinberg and the Guay enveloping embeddings to the context of spherical varieties?*

Note that both the theory of very flat monoids and the theory of very flat symmetric embeddings are defined over very concise varieties. By this we mean that the theory of very flat monoids studies embeddings of a groups with semisimple part simply-connected, while the theory of very flat symmetric embeddings studies embeddings of symmetric varieties with semisimple part equal to G/G^θ for G simply-connected and $\theta \in \mathrm{Aut}_2(G)$, or, equivalently, to G/G_0^θ, for any semisimple group G and $\theta \in \mathrm{Aut}_2(G)$. In both cases, the wonderful compactifications of G^{ad} and G/G_θ play an important role.

The above motivates the question of what groups associated to a spherical subgroup $H \subset G$ play the same role as G_0^θ and G_θ in the theory of symmetric varieties. For the role of G_θ it seems clear that one should pick the *spherical closure* $\bar{H}$ of H. This is the minimal group $\bar{H}$ containing H such that the space $G/\bar{H}$ admits a wonderful compactification. For some cases it looks like a good candidate H^0 to replace G_0^θ could be the common kernel of all the characters of H, but this does not work in every case since there are examples where the common kernel of the characters of G^θ is not equal to G_0^θ.

In any case, if one takes a semisimple group G and a spherical subgroup $H \subset G$, and finds $H^0 \subset H$ a good analog of G_0^θ, it makes sense to consider the category of simple affine embeddings of spherical homogeneous spaces with semisimple part G/H^0. Here, by the semisimple part of a homogeneous space G/H we simply mean the quotient $G'/(H \cap G')$.

For any such spherical embedding Σ of a homogeneous space O_Σ that has semisimple part equal to G/H^0, one can consider the torus $A_\Sigma = O_\Sigma/G$, and the GIT quotient $\mathbb{A}_\Sigma := \Sigma /\!\!/ G$. We call this $\mathbb{A}_\Sigma$ the *abelianization* of Σ and the natural projection $\Sigma \to \mathbb{A}_\Sigma$ the *abelianization map*. This allows to give a definition of *very flat spherical embedding of* G/H^0 and of the category $\mathcal{VF}(G/H^0)$ of very flat spherical embeddings with excellent morphisms.

The remaining problems would be to describe the objects of this category $\mathcal{VF}(G/H^0)$ in terms of their weight semigroups, and to find a universal object, if it exists. For the second problem, if we take H^0 equal to the common kernel of the characters of H, we expect from the results of Brion [Bri07] that the correct candidate is the *Brion–Cox embedding* of G/H^0.

We explain now the definition of the Brion–Cox embedding. If Σ is a smooth projective variety such that its Picard group $\mathrm{Pic}(\Sigma)$ is free, then one can consider its *Cox ring*

$$\mathrm{Cox}(\Sigma) = \bigoplus_{L \in \mathrm{Pic}(\Sigma)} H^0(X, L).$$

Wonderful varieties are well known to have a free Picard group. If G/H is a spherical homogeneous space, we define the *Brion–Cox embedding* to be the spectrum of the Cox ring of the wonderful compactification $\overline{G/\bar{H}}$ of $G/\bar{H}$, that is, we put

$$BC(G/H) = \mathrm{Spec}(\mathrm{Cox}(\overline{G/\bar{H}})).$$

We refer to [Bri07] and to [Tim11, Section 30.5] for more information about the Brion–Cox embedding. An important property of the Brion–Cox embedding is that, if G is a semisimple simply-connected group, regarded as a spherical G-variety, then $BC(G)$ is equal to the Vinberg monoid $\mathrm{Env}(G)$. However, if G/G^θ is a symmetric variety, the relation between $BC(G/G^\theta)$ and the Guay embedding $\mathrm{Env}(G/G^\theta)$ has not been yet dilucidated, to the best of our knowledge.

Question 3. *Let* G *be a semisimple simply-connected group and* $\theta \in \mathrm{Aut}_2(G)$ *an involution. What is the relation between the Brion–Cox embedding* $BC(G/G^\theta)$ *and Guay's envelopping embedding* $\mathrm{Env}(G/G^\theta)$*?*

To sum up, we think that a generalization of the theory of very flat monoids and very flat symmetric varieties should follow the next outline:

1. Determine the correct analogues of G_0^θ and G_θ.
2. Define the abelianization.
3. Determine the very flat objects in terms of their weight semigroup.
4. Find a (uni)versal very flat object.

Finally, in order to have a description of a conjectural Hitchin fibration for spherical varieties in terms of cameral covers, one needs to obtain some generalization of the results of Richardson concerning the invariant theory of a symmetric variety G/G^θ to spherical varieties. This would also be a "multiplicative analog" of the results of Knop [Kno94]. This is a problem still widely open and we refer to the comments given by Knop in a MathOverflow answer [Kno] for more information about it. Conjecturally, at least for H reductive, we expect that the GIT quotient $(G/H) /\!\!/ H$, for G/H a spherical homogeneous space, is isomorphic to $T_H/W_{G/H}$, for $T_H = T/(T \cap H)$, $T \subset G$ a maximal torus contained in a Borel subgroup $B \subset G$ with a dense orbit in G/H, and $W_{G/H}$ the *little Weyl group* of the spherical variety G/H, as defined by Brion [Bri90]. The results of Luna–Richardson [LR79] might give some insight into this problem.

INTRINSIC GAUGE EQUATIONS

The Charbonneau–Hurtubise–Smith theorem, our Theorem 3.2.13, can be understood as a Hitchin–Kobayashi correspondence for mini-holomorphic bundles, in the sense that it relates a polystability condition, defined at the (mini-)holomorphic level with a gauge theoretical equation (the HB equation) obtained from the moment map of the action of some group of gauge transformations on an infinite dimensional space of pairs. At the moduli space level, this implies that the moduli space of polystable mini-holomorphic bundles can be constructed as a symplectic (and, in fact, Kähler) quotient. Provided the equivalence of categories (for fixed $\mathbf{t}$) between mini-holomorphic bundles and multiplicative Higgs bundles, the CHS correspondence can in turn be understood as a Hitchin–Kobayashi correspondence for multiplicative Higgs bundles.

However, it still makes sense to pose the question of whether a stability condition, a gauge equation and a Hitchin–Kobayashi correspondence could be given in a completely intrinsic manner; that is, only considering data depending on the Riemann surface X, and not involving the circle S^1 nor the parameters $\mathbf{t}$, at least not a priori.

We expect that this problem fits inside the more general theory of pairs considered by Mundet i Riera in [MiR00]. These are pairs of the form (E, φ), where $E \to X$ is a holomorphic G-bundle over a compact Kähler manifold X, and φ is a holomorphic section of the associated bundle $E(\Sigma)$, for Σ a Kähler manifold endowed with a left action of G. As additional data, one needs to fix a maximal compact subgroup K, an invariant bilinear form (providing an isomorphism $\mathfrak{k} \cong \mathfrak{k}^*$), and suppose that there exists a moment map $\mu : \Sigma \to \mathfrak{k}$ for the induced K-action on Σ. Note that, given a reduction h of the structure group of E from G to K, the section φ can now also be regarded as a section of $E_h(\Sigma)$. Mundet i Riera defines a stability condition for this kind of pairs, depending on some central element $C \in Z(\mathfrak{k})$, and proves that a pair satisfies this condition if and only if the Chern connection ∇ of E associated to h gives a solution of the equation

$$F_\nabla + \mu(\varphi) = C.$$

The formulation of multiplicative Higgs bundles in terms of monoids, allows us to regard a multiplicative Higgs bundle of singularity type $(\mathbf{D}, \boldsymbol{\lambda})$ as a pair (E, φ), for $E \to X$ a holomorphic G-bundle and φ a holomorphic section of the associated bundle $E(M^{\lambda,0})$. Here, $M^{\lambda,0}$ is the open dense subvariety of the monoid M^{λ} defined in Section 2.1 (we refer to that same section for the rest of the notation). Moreover, to account for the $\mathbf{D}$ in the singularity type, we have to impose the condition that the abelianization of φ yields the element $(\mathcal{O}_X(\mathbf{D}), \mathbf{s})$.

This suggests that multiplicative Higgs bundles could indeed be pairs of the type considered by Mundet i Riera, assuming that the monoids M^{λ} have natural Kähler structures admitting moment maps for the action of K. To the best of our knowledge, Kähler structures and moment maps on these kind of monoids is a problem which is yet to be studied. Thus, we suggest the following.

Problem 2. *Study the Kähler structures on the monoids M^{λ}, if they exist, and the associated moment maps for the action of K on them, induced by the natural action of G.*

THE DE RHAM SIDE

One of the most important features of the theory of (additive) Higgs bundles is the *nonabelian Hodge correspondence*; this is a compendium of results of Corlette, Donaldson, Hitchin, Simpson, and others, [Cor88, Don87, Hit87a, Sim88] which gives a natural correspondence between isomorphism classes of polystable G-Higgs bundles over some Kähler manifold and conjugation classes of reductive representations of the fundamental group of the Kähler manifold into the group G. It is common in the literature of the topic, following the analogy with the Betti, de Rham and Dolbeault cohomology theories, to refer to the moduli space of Higgs bundles as the *Dolbeault moduli space*, while the moduli space of flat connections is the *de Rham moduli space* and the moduli space of representations is the *Betti moduli space*. The nonabelian Hodge correspondence can be understood as the fact that these three moduli spaces are diffeomorphic (in fact, real-analytically isomorphic). Although the equivalence between the de Rham and the Betti moduli spaces is complex-analytic, the equivalence between the de Rham and the Dolbeault spaces is not. This fact is deeply intertwined with the hyper-Kähler structure of the Hitchin moduli space. Indeed, if one considers the three complex structures I, J and K, the structure I can be regarded as the natural complex structure of the Dolbeault moduli space, while J and K can be understood as coming from the complex structure of the Betti moduli space. This is also closely related to the twistor space description introduced in Deligne's letters to Simpson. Moreover, as we have mentioned previously in this document, under the nonabelian Hodge correspondence Higgs bundles for symmetric pairs associated to an involution $\theta \in \mathrm{Aut}_2(G)$ are matched with representations of the fundamental group factoring through the real form $G_{\mathbb{R}}$ of G associated to θ.

As we explained in Section 3.3, when X is an elliptic curve, the moduli space of singular monopoles on $S^1 \times X$ of fixed type is also a hyper-Kähler manifold. Of the three complex structures I_t, I_x and I_y composing the hyper-Kähler structure, the first structure I_t is well understood to arise from mini-holomorphic bundles (or equivalently, from mulitplicative Higgs bundles), but it is not clear if the other two structures are natural complex structures corresponding to a moduli space of well understood geometric structures. A twistor description in this case is also not known (although we refer the reader to the paper of Elliott–Pestun [EP19] for the twistor description of a very similar problem).

Problem 3. *Study the twistor space associated to the moduli space of singular monopoles on $S^1 \times X$, for X an elliptic curve, and construct a "de Rham moduli space" parametrizing "well understood" geometric structures with complex structure given by I_x or, equivalently, by I_y or any $\mathbb{C}$-linear combination of I_x and I_y.*

Moreover, under such description of a "de Rham side" of the moduli space of monopoles, multiplicative (G, θ)-Higgs bundles should correspond to objects associated to the real form $G_{\mathbb{R}}$ associated to θ. In this direction, Jacques Hurtubise has suggested (in analogy with the analogous problem in the theory of Higgs bundles, and understanding that both theories arise from dimensional reductions of the self-dual Yang–Mills equations in $\mathbb{R}^4$) that, if (∇, Φ) is a singular monopole, one should consider the complex valued connections $\nabla_x + i\Phi$ and $\nabla_y - i\nabla_t$. To this respect, we just make the following comment. If we recall from the description of fixed monopoles (∇, Φ) under the involution ι_-^a given in Section 3.4, over the fixed surfaces $\{1\} \times X$ and $\{-1\} \times X$, the components ∇_x and ∇_y of a fixed monopole reduce to some $\mathfrak{k}^{\theta_0}$ and $\mathfrak{k}^{\theta_{2\pi}}$, while the components ∇_t and Φ reduce to $\mathfrak{m}_K^{\theta_0}$ and $\mathfrak{m}_K^{\theta_\pi}$. It is now clear then that, when we have those reductions, the operators $\nabla_x + i\Phi$ and $\nabla_y - i\nabla_t$ take values in the Lie algebra of the real forms $\mathfrak{g}_{\mathbb{R}}^0$ and $\mathfrak{g}_{\mathbb{R}}^\pi$ associated to θ_0 and θ_π, respectively.

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