An invitation to nonabelian Hodge theory

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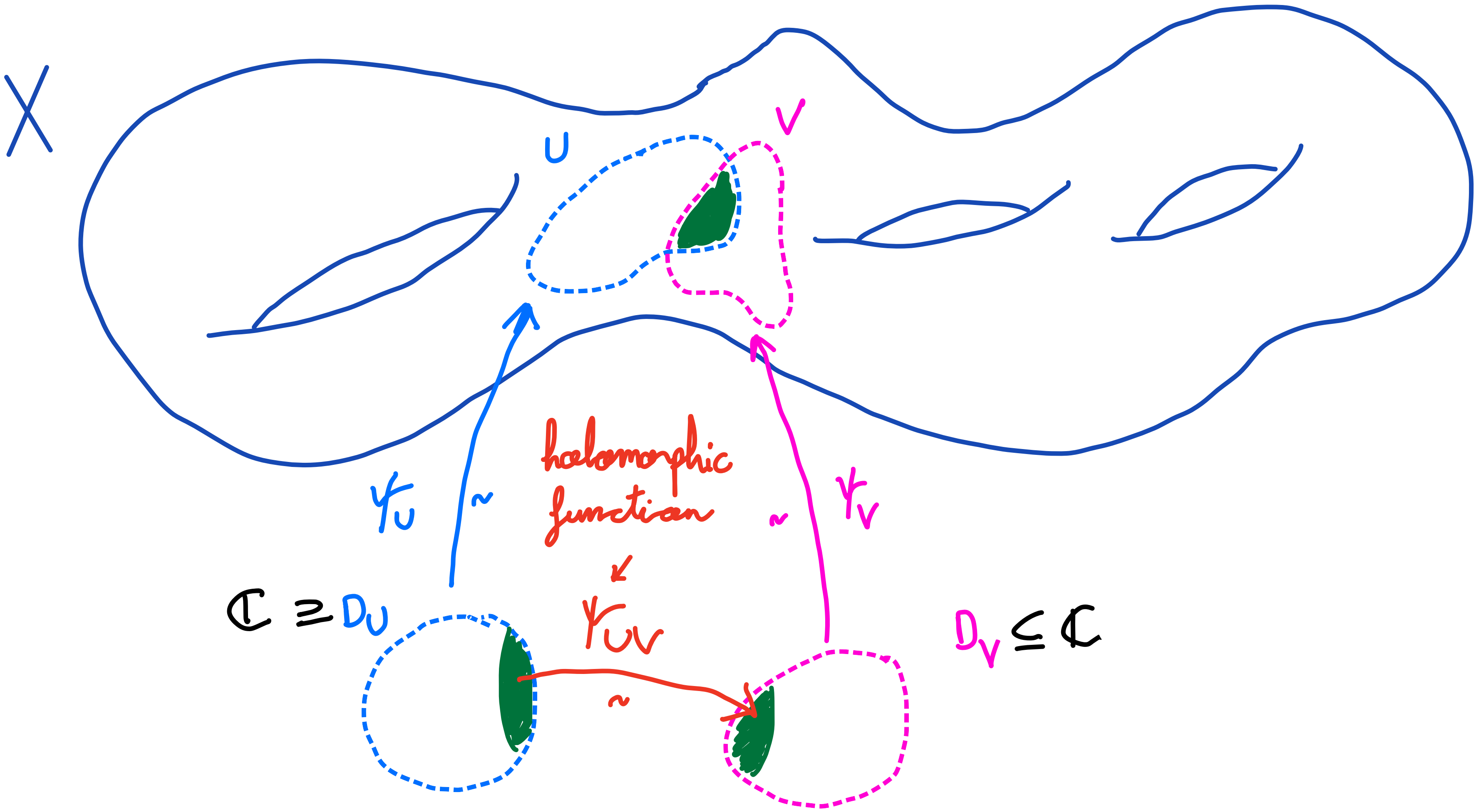
Guillermo Gallego
(FU Berlin)

Ingredient 1: Riemann Surfaces

= surface + cpx. structure



Ingredient 2: Holomorphic vector bundles

$$
\begin{tikzcd}
E \arrow[d, "p"'] & p^{-1}(U) \arrow[l, hook'] \arrow[r, "\varphi_U", "\sim"'] \arrow[d, "p|_{p^{-1}(U)}"'] & U \times \mathbb{C}^{r} \arrow[dl, "pr_U"] \\
X & U \arrow[l, hook']
\end{tikzcd}
$$

r ← rank

$$
\begin{tikzcd}
(U \cap V) \times \mathbb{C}^r \arrow[rr, bend left, "\varphi_{UV}"] \arrow[dr, "pr_U"'] & p^{-1}(U \cap V) \arrow[l, "\varphi_U"', "\sim"] \arrow[r, "\varphi_V", "\sim"'] \arrow[d] & (U \cap V) \times \mathbb{C}^r \arrow[dl, "pr_V"] \\
& U \cap V &
\end{tikzcd}
$$

$$\varphi_{UV}(x, \vec{v}) = (x, g_{UV}(x) \cdot v)$$

$$g_{UV} : X \xrightarrow[\text{holomorphic}]{} GL_r(\mathbb{C})$$

The original problem

Classify holomorphic vector bundles on a compact Riemann surface X.

(Topological classification: rank and degree)

First attempt: Čech cohomology

$$\check{H}^1(X, GL_r(\mathbb{C})_{\mathcal{O}}) = \frac{\{(g_{UV} : U \cap V \longrightarrow GL_r(\mathbb{C})) \mid g_{UV} \cdot g_{VW} = g_{UW}\}}{g_{UV} \sim f_U\, g_{UV} f_V, \quad (f_U : U \to \mathbb{C})}$$

$$\mathrm{Bun}_r(X) = \left\{ \begin{array}{c} \text{Rank } r \text{ holom.} \\ \text{vector bundles} \end{array} \right\} \Big/ \text{iso.} \xrightarrow{\sim} \check{H}^1(X, GL_r(\mathbb{C})_{\mathcal{O}})$$

$$E \to X \longmapsto (g_{UV})$$

$$\text{"glue patches" } E_U \cong U \times \mathbb{C}^r \longleftarrow\!\shortmid (g_{UV})$$

Line bundles: The Jacobian

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

$$0 \to H^1(X, 2\pi i\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, 2\pi i\mathbb{Z}) \xrightarrow{\int_X} 2\pi i\mathbb{Z}$$

$$H^1(X,\mathcal{O}_X^*) \xrightarrow{\deg} 2\pi i\mathbb{Z}$$

$$H^1(X,\mathcal{O}_X^*) = \mathrm{Pic}(X) \supset \mathrm{Jac}(X) \mapsto 0$$

$$\frac{\mathbb{R}^{2g}}{(2\pi\mathbb{Z})^{2g}} \cong \frac{H^1(X, i\mathbb{R})}{H^1(X, 2\pi i\mathbb{Z})} \cong \frac{H^1(X,\mathcal{O}_X)}{H^1(X, 2\pi i\mathbb{Z})} \cong \mathrm{Jac}(X)$$

Hodge

2g-dimensional torus

$$\mathrm{Pic}(X) \cong \mathrm{Jac}(X) \times \mathbb{Z}$$

Abelian Hodge Theory

The isomorphism $H^1(X, i\mathbb{R}) \cong H^1(X, \mathcal{O}_X) \cong H^{0,1}(X)$ follows

from:

Theorem (Hodge)

Let $\omega \in \Omega^2(X, \mathbb{C})$. There exists a solution $f : X \to \mathbb{C}$.

to $\boxed{\Delta f = \omega}$

iff. $\int_X \omega = 0$.

"Maxwell equation"
(U(1)-gauge theory)

Abelian Hodge Theory = Abelian Gauge Theory

Summing up, we obtain a diffeomorphism:

Holomorphic side

$$U(1)^{2g} \cong \mathrm{Hom}(\pi_1(X), U(1)) \cong \frac{H^1(X, i\mathbb{R})}{H^1(X, 2\pi i\mathbb{Z})} \cong \frac{H^1(X, \mathcal{O}_X)}{H^1(X, 2\pi i \mathbb{Z})} \cong \mathrm{Jac}(X)$$

Topological side

mediated by solving U(1)-Yang-Mills

Complex version

$$H^{0,1}(X) \xleftrightarrow{\text{serre dual}} H^{1,0}(X)$$

$$(\mathbb{C}^*)^{2g} \cong \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \cong \frac{H^1(X,\mathbb{C})}{H^1(X,2\pi i\mathbb{Z})} \overset{\text{diffeo.}}{\cong} \frac{\overbrace{H^1(X,\mathcal{O}_X)}^{H^{0,1}(X)}}{H^1(X,2\pi i\mathbb{Z})} \oplus \overbrace{H^0(X,K_X)}^{H^{1,0}(X)} \cong T^*\mathrm{Jac}(X)$$

$(\mathbb{C}^*)^{2g}$ is diffeomorphic to $T^*\mathrm{Jac}(X)$

$(\mathbb{C}^*)^{2g}$ is not biholomorphic to $T^*\mathrm{Jac}(X)$

Issues in higher rank

$H^1(X, \mathcal{O}_X^*)$ ← abelian group, well behaved

$H^1(X, GL_r(\mathbb{C})_{\mathcal{O}})$ ← bad behaved

→ not a group

→ bad geometry

→ not separated

→ not of finite type

("jumping phenomena")

Classifications in low genus

- Grothendieck (1957) [Actually, Birkhoff (1909)]

  $E \to \mathbb{CP}^1$ splits as $E \cong \mathcal{O}(n_1) \oplus \ldots \oplus \mathcal{O}(n_r)$.

- Atiyah (1957)

  Classification of vector bundles on elliptic curves

Narasimhan-Seshadri (1965)

- Stability condition for vector bundles.

$$\mathrm{Bun}_r^s(X) \underset{\text{open}}{\subseteq} \mathrm{Bun}_r(X)$$

Theorem (Narasimhan - Seshadri)

There is a homeomorphism

$$\mathrm{Bun}_{r,0}^s(X) \cong \mathrm{Hom}(\pi_1(X), U(n)) / U(n)$$

deg. 0

Flat connections 1

- **Connections**: $d_A : \Gamma(X,E) \longrightarrow \Omega^1(X,E)$

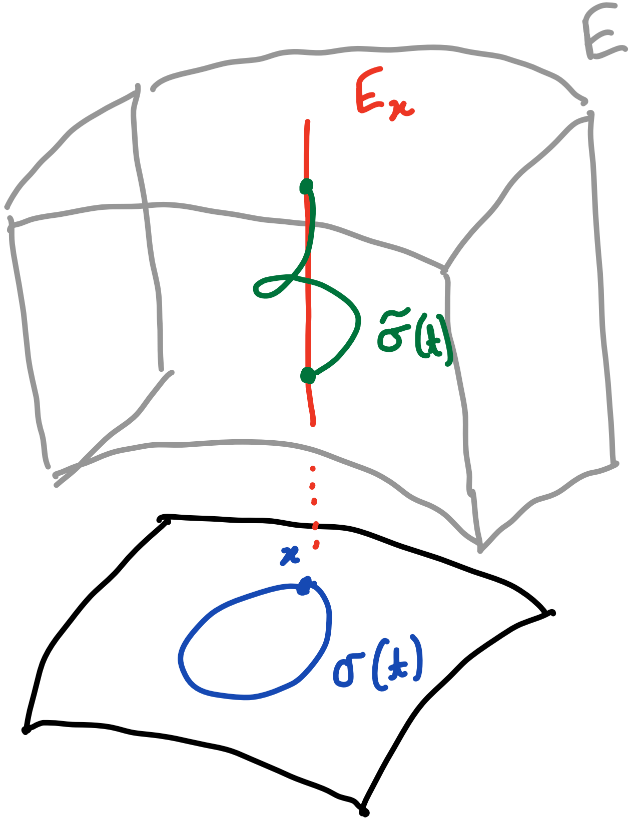
  $d_A(f\cdot s) = df\cdot s + f\, d_A s$ (Leibniz rule)

  Extend: $\Gamma(X,E) \xrightarrow{d_A} \Omega^1(X,E) \xrightarrow{d_A} \Omega^2(X,E)$

- **Curvature**: $F_A = d_A \circ d_A \in \Omega^2(X, \mathrm{End}\, E)$.

  A is *flat* if $\boxed{F_A = 0}$.

Flat connections 2. Holonomy



$\tilde\sigma(t)$ solution of $\quad d_A\tilde\sigma(t)\big(\dot\sigma(t)\big) = 0\,.$

$$\tilde\sigma(1) = A(\sigma)\cdot\tilde\sigma(0)$$

$$\begin{tikzcd}
A : \{\text{loops at } x\} \arrow[r] \arrow[d] & GL(E_x) \cong GL_r(\mathbb{C}) \\
\pi_1(X,x) \arrow[ur, "A"'] &
\end{tikzcd}$$

$\boxed{\text{If } F \text{ is flat!}}$

Narasimhan-Seshadri, reinterpreted (Atiyah-Bott, 1980)

$$\mathrm{Bun}^s_{r,0}(X) \cong \{\text{flat } U(n)\text{-bundles}\}/\text{gauge}$$

<u>Theorem</u> (Narasimhan - Seshadri as reproved by Donaldson, 1983)

Any <u>stable</u> holomorphic vector bundle $E \to X$ admits a Hermitian metric H with

curvature of Chern conn. A_H

$$\boxed{F_H = \lambda\,\omega_X}, \quad \lambda = \text{cte}.$$

fixed 2-form on X

[equivalent to U(n)-Yang-Mills].

Splitting complex connections

- $\mathbb{E} \to X$ smooth complex vector bundle of rank r
- H Hermitian metric on $\mathbb{E}$. $\mathfrak{u}_H\mathbb{E} \leftarrow$ skew-Hermitian endomorphisms of E
- D complex connection on $\mathbb{E}$

$\operatorname{End}\mathbb{E} = \mathfrak{u}_H\mathbb{E} + i\,\mathfrak{u}_H\mathbb{E}$

$$D = \nabla + i\,\Phi$$

section of $\mathfrak{u}_H\mathbb{E}$ (Higgs field)

H-unitary connection

Splitting complex connections. Flatness

- $D = \nabla + i\Phi$ as before.
- $F_D = F_\nabla + \Phi \wedge \Phi + i\nabla\Phi$.

$$F_D = 0 \iff \left| \begin{array}{l} F_\nabla + \Phi \wedge \Phi = 0. \\ \nabla\Phi = 0. \end{array} \right.$$

A coordinate change

$\nabla = \nabla^{1,0} + \nabla^{0,1} \rightsquigarrow \bar\partial_E = \nabla^{0,1}$ holomorphic structure on $\mathbb{E}$.

$\Omega^1(X, \mathfrak{u}_H\mathbb{E}) \quad \Omega^{1,0}(X, \mathrm{End}\,\mathbb{E}) \quad \Omega^{0,1}(X, \mathrm{End}\,\mathbb{E})$

$\Phi = \Phi^{1,0} + \Phi^{0,1} \rightsquigarrow \varphi = \Phi^{1,0}$

$$\boxed{G_H = \bar\partial_E \varphi}$$

(pseudocurvature)

- If $G_H = 0$, then φ is a <u>holomorphic</u> section of $E = (\mathbb{E}, \bar\partial_E)$.

The Donaldson-Corlette theorem

- H is a <u>harmonic metric</u> if $\boxed{G_H = 0}$.

<u>Theorem</u> (Donaldson, 1987; Corlette, 1988)

$(\mathbb{E}, D)$ induces a semisimple representation $\rho : \pi_0(X) \longrightarrow GL_r(\mathbb{C})$

if and only if there exists a unique H on $\mathbb{E}$ with $G_H = 0$.

Higgs bundles

- A Higgs bundle on X is a pair (E, φ)
  - $E \to X$ holomorphic vector bundle.
  - $\varphi \in \Gamma(X, E)$ holomorphic section.

Donaldson-Corlette reinterpreted

$$\mathrm{Hom}(\pi_1(X), GL_r(\mathbb{C}))^{ss} / GL_r(\mathbb{C}) \xrightarrow{\sim} \frac{\{\text{flat bundles}\}}{\text{gauge}} \longrightarrow \frac{\{\text{Higgs bundles}\}}{\text{iso.}}$$

$$(\mathbb{E}, D) \longmapsto (E, \varphi)$$

Find harmonic metric H

The Hitchin equations

- Stability condition for Higgs bundles $\mathrm{Higgs}_r(X)^s \underset{\text{open}}{\subseteq} \mathrm{Higgs}_r(X)$.

Theorem (Hitchin, 1987; Simpson, 1988)

A Higgs bundle (E, φ) is stable if and only if it admits a Hermitian metric H such that

constant

$$\boxed{F_H + [\varphi, \varphi^*] = \lambda}.$$

$\boxed{\text{The Hitchin equations}}$

Nonabelian Hodge theory

$$\begin{tikzcd}
\left\{ \begin{array}{c} \text{(semisimple)} \\ \text{flat bundles} \\ \text{of rk. } r \end{array} \right\} \arrow[d] & \left\{ \begin{array}{l} \text{Solutions to} \\ \left| \begin{array}{l} F_\nabla + \Phi \wedge \Phi = 0 \\ \nabla \Phi = \nabla_* \Phi = 0 \end{array} \right. \end{array} \right\} \arrow[d] & \left\{ \begin{array}{c} \text{(stable)} \\ \text{Higgs bundles} \end{array} \right\} \arrow[d] \\
\mathrm{Flat}_r(X)^{ss} \arrow[r, leftrightarrow, "\text{Donaldson–Corlette}"'] \arrow[d, leftrightarrow] & \mathrm{Hitch}_r(X) \arrow[r, leftrightarrow, "\text{Hitchin–Simpson}"'] & \mathrm{Higgs}_r(X)^s \\
\mathrm{Hom}(\pi_1(X), GL_r(\mathbb{C}))^{ss} / GL_r(\mathbb{C}) & &
\end{tikzcd}$$

$$\boxed{\mathrm{Flat}_r(X)^{ss} \xrightarrow[\text{homeo.}]{} \mathrm{Higgs}_r(X)^s}$$

Hyperkähler structure

$$\mathcal{M} = \mathrm{Hitch}_r(X) = \left\{ (\nabla, \Phi) \;\middle|\; \begin{array}{l} F_\nabla + \Phi \wedge \Phi = 0 \\ \nabla \Phi = \nabla * \Phi = 0 \end{array} \right\}$$

→ Riemannian metric: $g((A_1, \Phi_1), (A_2, \Phi_2)) = -\int_X \mathrm{tr}(A_1 \wedge * A_2 + \Phi_1 \wedge * \Phi_2)$

→ Complex structures:

$$\left| \begin{array}{l} I_1(A, \Phi) = (*A, - * \Phi) \\ I_2(A, \Phi) = (-\Phi, A). \end{array} \right. \qquad (I_3 = I_1 \cdot I_2)$$

$(\mathcal{M}, g, I_1, I_2, I_3)$ is a hyperkähler manifold

Hyperkähler structure

$(\mathcal{M}, g, I_1, I_2, I_3)$ is a hyperkähler manifold

$(\mathcal{M}, g, I_1) \cong \mathrm{Higgs}_r(X)^s$ (Hitchin - Simpson)

$(\mathcal{M}, g, I_2) \cong \mathrm{Flat}_r(X)^{ss}$ (Donaldson - Corlette)

The Hitchin system

$$h : \mathrm{Higgs}_r(X) \longrightarrow \mathcal{A}_r = \bigoplus_{i=1}^{r} H^0(X, (\Omega^1_X)^{\otimes r})$$

$$(E, \varphi) \longmapsto (a_1(\varphi), \dots, a_r(\varphi))$$

$$\det(T\,\mathbb{I}_n - \varphi) = T^n + a_1(\varphi)T^{n-1} + \dots + a_n(\varphi)$$

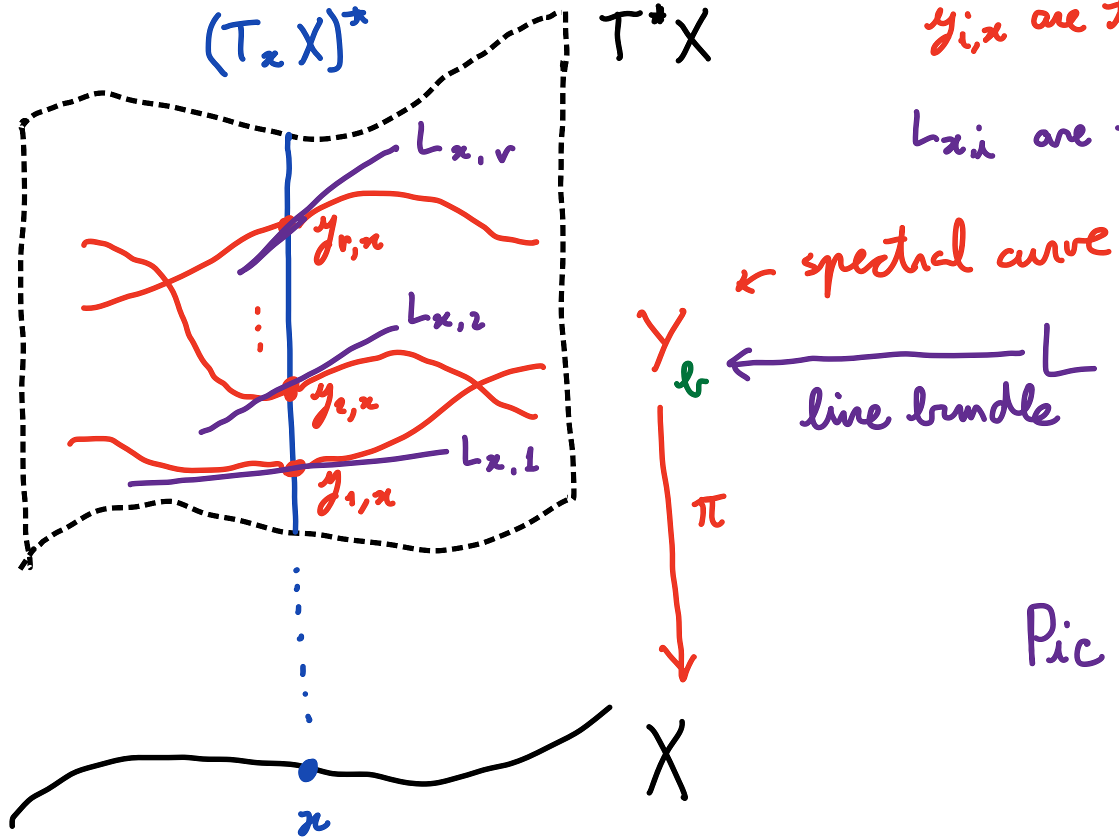
<u>Theorem</u> (Hitchin) h determines an algebraically integrable system.

The spectral correspondence

$b = h(E, \varphi)$

$y_{i,x}$ are the eigenvalues of $\varphi(x)$

$L_{x,i}$ are the eigenspaces



$$\mathrm{Pic}(Y_b) \xrightarrow[\sim]{\pi_*} h^{-1}(b)$$

SYZ Mirror Symmetry

$$\left\{(E,\varphi) \;\middle|\; \begin{array}{l}\det E \cong \mathcal{O}_X \\ \operatorname{tr}\varphi = 0\end{array}\right\} \cong \mathrm{Higgs}_{SL_r(\mathbb{C})}(X) \qquad\qquad \mathrm{Higgs}_{PGL_r(\mathbb{C})}(X) \cong \mathrm{Higgs}_r(X)/\mathrm{Jac}(X)[r]$$

$$\begin{tikzcd}
\mathrm{Higgs}_{SL_r(\mathbb{C})}(X) \arrow[dr, "h_{SL_r}"'] & & \mathrm{Higgs}_{PGL_r(\mathbb{C})}(X) \arrow[dl, "h_{PGL_r}"] \\
& \mathcal{A}_r \ni b &
\end{tikzcd}$$

__Theorem__ (Hausel-Thaddeus, 2003)

$h_{SL_r}^{-1}(b)$ and $h_{PGL_r}^{-1}(b)$ are <u>dual abelian varieties</u>.

Homological Mirror Symmetry

- SYZ duality on Higgs $\overset{\text{NAHT}}{\rightsquigarrow}$

$$\begin{tikzcd} \mathrm{Flat}_{SL_r} \arrow[dr] & & \mathrm{Flat}_{PGL_r} \arrow[dl] \\ & \mathcal{A}_r & \end{tikzcd}$$

dual special Lagrangian fibrations

$\rightsquigarrow$ mirror Calabi-Yaus

$$\begin{tikzcd} \text{A-branes} \arrow[d, phantom, "\wr"] & & \text{B-branes} \arrow[d, phantom, "\wr"] \\ \mathrm{Fuk}(\mathrm{Flat}_{SL_r}) \arrow[rr, leftrightarrow, "\text{Homological mirror symmetry}", "\sim"'] \arrow[d, leftrightarrow, "\sim"'] & & \mathrm{Coh}(\mathrm{Flat}_{PGL_r}) \arrow[dll, leftrightarrow, bend left, "\sim"] \\ \text{D-Mod}(\mathrm{Bun}_{SL_r}) & & \end{tikzcd}$$

(Kapustin-Witten, 2007)

Geometric Langlands Conjecture

(Beilinson-Drinfeld, ~1990's)

Thanks for your attention