Monoids, symmetric varieties and multiplicative Hitchin systems

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Linear algebraic groups

- Let k be an algebraically closed field of characteristic 0.
- A group scheme G over k is a group object in the category of k-schemes. That is, the functor of points $\underline{G} = \mathrm{Maps}(-, G)$ from Sch_k to Sets factors through the category of groups. (Equivalently, this means that there are morphisms $m : G \times G \longrightarrow G$, $e : \mathrm{Spec}(k) \longrightarrow G$ and $(-)^{-1} : G \longrightarrow G$ working as multiplication, identity element and inverse).
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- A linear algebraic group over k is a smooth and affine group shceme over k of finite type.
- Example:

$$\begin{aligned} \mathrm{GL}_n : \mathrm{Sch}_k &\longrightarrow \mathrm{Groups} \\ \mathrm{Spec}(R) &\longmapsto \mathrm{GL}_n(R). \end{aligned}$$

  We write $\mathbb{G}_m = \mathrm{GL}_1$.
- In fact, every linear algebraic group admits a closed embedding

$$i : G \hookrightarrow \mathrm{GL}_n,$$

  for some n.

Reductive groups

- For several reasons, we like to work with groups which are reductive.
- A reductive group can always be decomposed as $G = G'Z$, where G is a semisimple group and Z is an algebraic torus.
- An algebraic torus T is a group scheme over k isomorphic to $\mathbb{G}_m^r$, for some r.
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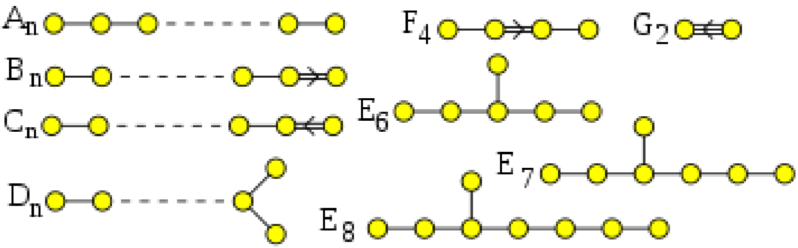
- A semisimple group can always be decomposed (up to isogeny) as a product of simple groups.
- Simple groups are classified (up to isogeny) by Dynkin diagrams (that is, we get the classical groups SL_n, SO_n, Sp_n, for types A, B, C and D and the exceptional groups for types F, E and G):



Monoids

- A monoid (in the abstract sense) is a set with an associative multiplication with neutral element.
- An algebraic monoid M over k is a group object in the category of k-schemes. Equivalently, this means that there is a multiplication morphism $m : G \times G \longrightarrow G$ with neutral element $e : \mathrm{Spec}(k) \longrightarrow G$, but not necessarily an inverse.

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- The set of invertible elements (units) of a monoid is a group. In the algebraic category we get a closed subscheme $M^\times \hookrightarrow M$ which is a group scheme.
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- Example:

$$\begin{aligned} \mathrm{Mat}_n : \mathrm{Sch}_k &\longrightarrow \mathrm{Monoids} \\ \mathrm{Spec}(R) &\longmapsto \mathrm{Mat}_{n\times n}(R). \end{aligned}$$

- From results of Rittatore it follows that monoids with affine unit group are affine, and that these are the same as equivariant (under left and right multiplication) embeddings of its unit group.

Abelianization

- Let M be a reductive monoid and $M^\times$ its unit group. We can split $M^\times = GZ$ for G a semisimple group and Z an algebraic torus.
- The abelianization of M is the scheme

$$\mathbb{A}_M = M /\!\!/ (G \times G) = \mathrm{Spec}(k[M]^{G \times G}).$$

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- The natural quotient map $\alpha_M : M \longrightarrow \mathbb{A}_M$ is called the abelianization map.
- The abelianization $\mathbb{A}_M$ can be understood as a toric variety for the algebraic torus $A_M = Z/(Z \cap G)$. Indeed, $\mathbb{A}_M$ is the toric variety with weight semigroup

$$P_+(\mathbb{A}_M) = P_+(M) \cap \mathbb{X}^*(A_M),$$

where, if $T_M \subset M^\times$ is a maximal torus $P_+(M) \subset \mathbb{X}^*(T_M)$ is such that

$$k[M] = \bigoplus_{\chi \in P_+(M)} V_\chi^* \otimes V_\chi,$$

given that $k[M]$ is a $(M^\times \times M^\times)$-submodule of $k[M^\times]$.

Very flat monoids

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- A reductive monoid M is very flat if the abelianization map $\alpha_M : M \longrightarrow \mathbb{A}_M$ is dominant, flat and with integral fibres.
- Vinberg proved that given a semisimple simply-connected group G there exists a special very flat monoid $\mathrm{Env}(G)$ with $(\mathrm{Env}(G)^\times)' = G$ such that every other very flat monoid M with $(M^\times)' = G$ can be obtained as a pullback

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\alpha_M"] & \mathrm{Env}(G) \arrow[d, "\alpha_{\mathrm{Env}(G)}"] \\ \mathbb{A}_M \arrow[r] & \mathbb{A}_{\mathrm{Env}(G)} . \end{tikzcd}$$

- The monoid $\mathrm{Env}(G)$ is called the Vinberg (enveloping) monoid.

The Vinberg monoid

- Let G be a semisimple simply-connected group, $T \subset G$ a maximal torus and $G_+ := (G \times T)/Z_G$.
- Let $\alpha_1, \dots, \alpha_r$ be the simple roots and $\varpi_1, \dots, \varpi_r$ the dominant weights of G
- Consider the map

$$\begin{aligned} G_+ &\longrightarrow \Big(\bigoplus_{i=1}^{r} \mathrm{End}(V_{\varpi_i}) \Big) \times \mathbb{A}^r \\ [g,t] &\longmapsto ((t^{\varpi_i} g)_{i=1}^r, (t^{\alpha_1}, \dots, t^{\alpha_r})). \end{aligned}$$

- $\mathrm{Env}(G)$ is defined as the closure of the image of the above map.
- $A_{\mathrm{Env}(G)} = T^{\mathrm{ad}}$ and $P_+(\mathbb{A}_{\mathrm{Env}(G)}) = \mathbb{Z}_+\langle \alpha_1, \dots, \alpha_r \rangle$.

Vinberg's classification

- Let M be a very flat monoid with $\mathbb{A}_M$ isomorphic to $\mathbb{A}^n$ for some n. Pick an isomorphism $\mathbb{G}_m^n \to A_M$.
- Under the above isomorphism, any map $A_M \to T^{\mathrm{ad}}$ is of the form

$$\begin{aligned} \boldsymbol{\lambda} : \mathbb{G}_m^n &\longrightarrow T^{\mathrm{ad}} \\ (z_1, \dots, z_n) &\longmapsto z_1^{\lambda_1} \cdots z_n^{\lambda_n}, \end{aligned}$$

with the $\lambda_i \in \mathbb{X}_*(T^{\mathrm{ad}})$.

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  with the $\lambda_i \in \mathbb{X}_*(T^{\mathrm{ad}})$.
- The above map extends to a morphism $\boldsymbol{\lambda} : \mathbb{A}^n \longrightarrow \mathbb{A}_{\mathrm{Env}(G)}$ if and only if the λ_i are dominant.
- Thus, every very flat monoid with $\mathbb{A}_M$ isomorphic to $\mathbb{A}^n$ is of the form

$$M^{\boldsymbol{\lambda}} = \mathrm{Env}(G) \times_{\boldsymbol{\lambda}, \mathbb{A}_{\mathrm{Env}(G)}} \mathbb{A}^n,$$

  for some $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ with the $\lambda_i \in \mathbb{X}_*(T^{\mathrm{ad}})_+$.

Formal loop parametrization (1)

- Let $\mathcal{O} = k[[z]]$ and $F = k((z))$ be the ring of formal power series and the field of formal Laurent series in the formal variable z, respectively. We denote by $G(\mathcal{O}) = \mathrm{Maps}(\mathrm{Spec}(\mathcal{O}), G)$ and $G(F) = \mathrm{Maps}(\mathrm{Spec}(F), G)$ the formal arc group and the formal loop group, respectively.
- The formal loop group decomposes as

$$G(F) = \bigsqcup_{\lambda \in \mathbb{X}_*(T)_+} G(\mathcal{O}) z^{\lambda} G(\mathcal{O}),$$

for z^{λ} the image of z under $\lambda(F) : F^{\times} \longrightarrow T(F)$.

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for z^λ the image of z under $\lambda(F) : F^\times \longrightarrow T(F)$. Moreover, the closure of an orbit $G(\mathcal{O})z^\lambda G(\mathcal{O})$ is equal to

$$\overline{G(\mathcal{O}) z^\lambda G(\mathcal{O})} = \bigsqcup_{\mu \leq \lambda} G(\mathcal{O}) z^\mu G(\mathcal{O}),$$

where we say that $\mu \leq \lambda$ if and only if $\lambda - \mu \in \mathbb{Z}_+\langle \alpha_1^\vee, \dots, \alpha_r^\vee \rangle$.

Formal loop parametrization (2)

- The decomposition of $G(F)$ can be extended to a stratification

$$T^{\mathrm{ad}}(F) \cap \mathbb{A}_{\mathrm{Env}(G)}(\mathcal{O}) = \bigsqcup_{\lambda \in \mathbb{X}^*(T^{\mathrm{ad}})_+} T^{\mathrm{ad}}(\mathcal{O}) z^{\lambda}.$$

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- Consider the fibered product

$$\mathrm{Env}^{\lambda}(G) = (T^{\mathrm{ad}} \times \mathrm{Env}(G)) \times_{z^{\lambda}, \mathbb{A}_{\mathrm{Env}(G)}} \mathrm{Spec}(\mathcal{O}).$$

- We obtain a stratification

$$\mathrm{Env}(G)(\mathcal{O}) \cap G_+(F) = \bigsqcup_{\lambda \in \mathbb{X}^*(T^{\mathrm{ad}})_+} \mathrm{Env}^{\lambda}(G)(\mathcal{O}).$$

- It is a result of J. Chi that for any $\phi \in G_+(F)$, we have $\phi \in \mathrm{Env}^{\lambda}(G)(\mathcal{O})$ if and only if the image of ϕ in $G^{\mathrm{ad}}(F)$ belongs to $\overline{G(\mathcal{O}) z^{\lambda} G(\mathcal{O})}$.

Invariant theory for a very flat monoid (Vinberg)

- It is a classical result of Chevalley that

$$k[G]^G = k[T]^W = k[a_1, \dots, a_r],$$

with $a_i = \mathrm{tr}(\rho_i)$. Equivalently,

$$G /\!\!/ G = T/W \cong \mathbb{A}^r,$$

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  for W the Weyl group of G.
- If M is a very flat monoid with $(M^\times)' = G$, we can extend the above to get

$$M /\!\!/ G = T/W \times \mathbb{A}_M.$$

The Hitchin fibration associated to a very flat monoid (Bouthier–Chi–Wang)

- Recall that if M is a G-scheme, by definition the quotient stack $[M/G]$ sends any test scheme S to the groupoid of pairs (E, φ), with $E \longrightarrow S$ a G-torsor and φ a section of the associated bundle $E \times_G M$.

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- If M is a very flat monoid with $(M^{\times})' = G$, we can consider the following sequence of stacks

$$[M/(G \times A_M)] \longrightarrow [(M /\!\!/ G)/A_M] \longrightarrow [\mathbb{A}_M/A_M] \longrightarrow \mathbb{B}A_M.$$

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- Let X be a smooth algebraic curve over k.
- Pulling the above sequence back to X, we obtain the Hitchin fibration associated to M.

Geometric interpretation (Bouthier–Chi–Wang)

- Suppose that $A_M = \mathbb{G}_m^n$, $\mathbb{A}_M = \mathbb{A}^n$ and $M = M^{\boldsymbol{\lambda}}$.
- $\mathrm{Maps}(X, \mathbb{B}A_M) = \mathrm{Pic}(X)^n$. Given $\boldsymbol{D} = (D_1, \ldots, D_n) \in \mathrm{Pic}(X)^n$, the composition $X \longrightarrow \mathbb{B}T^{\mathrm{ad}}$ is the $\mathbb{X}_*(T)$-valued divisor
$$\boldsymbol{\lambda} \cdot \boldsymbol{D} = \lambda_1 D_1 + \cdots + \lambda_n D_n.$$
- The fibre over $\boldsymbol{D}$ of the set $\mathrm{Maps}(X, [\mathbb{A}_M / A_M])$ is the space of sections $\bigoplus_{i=1}^n H^0(X, \mathcal{O}(D_i))$.

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- The fibre over $\boldsymbol{D}$ of the set $\mathrm{Maps}(X, [\mathbb{A}_M/A_M])$ is the space of sections $\bigoplus_{i=1}^n H^0(X, \mathcal{O}(D_i))$.
- The composition of a map $X \longrightarrow [M/(G \times A_M)]$ with the natural map $M \longrightarrow \mathrm{Env}(G)$ is a pair (E, φ) with $E \longrightarrow X$ a G-torsor and φ a section of $E \times_G \mathrm{Env}(G)$.
- Given a point $x \in X$, restricting φ to a formal disk $\mathrm{Spec}(\hat{\mathcal{O}}_{X,x})$ amounts to give an element of $\mathrm{Env}(G)(\mathcal{O}) \cap G_+(F)$. It is easy to see that this element lies precisely in $\mathrm{Env}^{\lambda_x}(\mathcal{O})$, for λ_x the coefficient of $\boldsymbol{\lambda} \cdot \boldsymbol{D}$ in x.
- Therefore, φ can be regarded as a "meromorphic" section of $E \times_G G$, with singularities in $\mathrm{Supp}(\boldsymbol{D})$ meromorphic datum $\mathrm{inv}(\varphi) \leq \boldsymbol{\lambda} \cdot \boldsymbol{D}$.

The multiplicative Hitchin fibration (Hurtubise–Markman & Bouthier–Chi–Wang)

Definition

A multiplicative Higgs bundle over X is a pair (E, φ) formed by a principal G-bundle $E \longrightarrow X$ and a section $\varphi \in \Gamma(X', E \times_G G)$, for $X' \subset X$ the complement of a finite subset.

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The multiplicative Hitchin fibration is the following map

$$(E, \varphi) \longmapsto (a_1(\varphi), \dots, a_r(\varphi)),$$

where we recall that the $a_i = \mathrm{tr}(\rho_i)$ were the generators of the invariant polynomial ring $k[G]^G$.

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- We have seen that the multiplicative Hitchin fibration can be recovered in terms of the Hitchin fibration for a very flat monoid.

Symmetric varieties (G.–García-Prada)

- In my thesis (supervised by O. García-Prada), I have considered pairs (E, φ) formed by a principal G^θ-bundle $E \longrightarrow X$ and a section φ of $E \times_{G^\theta} G/G^\theta$, where $\theta \in \mathrm{Aut}_2(G)$ is an involution.
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- Vinberg's theory of very flat monoids gets replaced by Nicolas Guay's theory of very flat embeddings of symmetric varieties.
- The Vinberg monoid is replaced by Guay's enveloping embedding.
- Question: Can this theory be generalized to other (more general) homogeneous spaces? Maybe to spherical varieties?

Thanks for your attention!